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LOCATION-TRANSPORT RELATIONSHIPS: STATE-OF-THE-ART, UNIFYING EFFORTS AND FUTURE DEVELOPMENTS

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Abstract

The aim of this study is to make a feasibility analysis of studies on location and transport interrelations.

The study consists of the following:

a. a review of the current state-of-the-art;

b. an identification of the directions of research emerging in this field;

c. an attempt to define the most promising aspects of research on which future efforts should be concentrated.

Key words: state-of-the-art of location-transport interrelations; demand analysis in location-transport studies; price formation in location-transport systems; the dynamics of location-transport interrelations; future research developments on location-transport interrelations.

CONTENTS

1. Introduction

1.1. *The aim and nature of the study*

The aim of this study is to make an analysis of studies involving location and transport interrelations and to assess the most important directions for future research in this field.

Our intention is to:

a. describe the state-of-the-art of work on transport-location interrelations;
b. identify the main directions in which this work is advancing;
c. look at the most promising avenues of research, with the aim of establishing how research efforts can be most effectively channelled in the future.

The interrelationships of location and transport are highly complex and consequently any description of the phenomena involved is necessarily non-univocal and cannot claim to be exhaustive. Our classification of studies is clearly not the only one possible, but appears to correspond by and large to the main themes emerging from research in this field (and from the authors' experience seems the most useful). We maintain, in any case, that other classifications are likely to differ only marginally. The categories we have adopted for the purpose of this study are therefore:

a. interrelations between location of economic activities and commodity flows;
b. interrelations between services and the journeys generated by their use;
c. interrelations between residential location and journeys to work;
d. interrelations between location and transport in the urban system;
e. interrelations between urban form and transport.

There are clearly a large number of theoretical, methodological and practical problems involved in the exploration of these interrelationships. Later in the study we describe the new trends emerging in this field. It is possible however, at the outset, to identify the three main streams which appear to offer the most promising subjects of research for the future:

a. models of spatial choice behaviour;
b. mechanisms of the dynamics and evolution of location-transport systems;
c. the various economic paradigms applied to the analysis of location-transport interrelationships.

1.2. *The most important existing approaches*

The obligatory starting point for a study aiming to describe the most important contributions to the understanding of the interrelationships between location and transport, is that of classic urban economics of the 19th century and in particular that emanating from the German school. More precisely, we must go back to von Thünen (1826) and later Weber (1909), Hoover (1948) and Isard (1956, pages 77-142 and 221-287), who wrote specifically about the interrelationship of location and transport and to Christaller (1933), Palander (1935), Lösch (1940) and more recently Isard (1956), Lefeber (1958) and Greenhut (1963) who have contributed to the question of a general location theory which implicity takes into account transport costs as well.

The most natural 'descendant' of this line of research is Beckmann (1968) (*) who managed to systematically combine geographical theory and urban economics with the methods and techniques which Operational Research had made available from the late 1940's on.

It was in this way that in the following decades geographic theory and urban economics forged ahead thanks to the work of, among others, Wingo (1961), Alonso (1964a) (**), Richardson (1969a, 1969b, 1973a, 1973c, 1977b, 1978) (***), Papageorgiou (ed.) (1976a) (****), Fujita (1978) (*****), Puu (1978, 1979b, 1981a, 1981b, 1981c, 1982a (******) and Kanemoto (1980b) (*******).

(*) Also Beckmann (1952, 1953, 1955, 1957b, 1958, 1969, 1970, 1971, 1972, 1973a, 1973b, 1974, 1975, 1976a, 1976b, 1981a, 1981b) and Beckmann and Marschak (1955), Beckmann and McPherson (1970), Beckmann and Schramm (1972), Beckmann and Buttler (1980), Golob and Beckmann (1971), Golob, Gustafsson and Beckmann (1973) and Koopmans and Beckmann (1957).

(**) Also Alonso (1960, 1964b, 1967, 1971).

(***) Also Richardson (1973b, 1977a).

(****) Also Papageorgiou (1971, 1976c, 1978, 1979, 1980, 1982, 1983) and Papageorgiou and Mullally (1976), Papageorgiou and Thisse (1982), Papageorgiou and Smith (1983), Casetti and Papageorgiou (1971) and Smith and Papageorgiou (1982).

(*****) Also Fujita (1975, 1976a, 1976b, 1979, 1980, 1981a, 1981b, 1981c, 1982a, 1982c, 1983) and Fujita and Kashiwadani (1976, 1982), Fujita and Ogawa (1982) and Ogawa and Fujita (1979, 1980a, 1980b).

(******) Also Puu (1977, 1979a, 1982b, 1983).

(*******) Also Kanemoto (1976, 1980a).

From the 1960's onwards there was alongside this a powerful development of mathematical models. Unlike the types of study referred to above which were based on specific interpretative paradigms, usually from economics, this branch had its roots in a quantitative formulations of empirical regularities which like social physics, spatial interaction and gravity models did not have economic underpinning (at least at the beginning).

It seems that the development of models has occurred partially with the aim of building and testing tools for direct application in town planning and partially with the aim of proposing an alternative to the strictly neo-classic approaches.

The models constructed for planning purposes were inspired initially by the pioneering work of Hansen (1959) and found their mainstay in the Lowry model (1964) (*) (**). Lowry's model itself was in fact immediately applied and extended. Among the early modifications were those of Crecine (1964), Brotchie (1965), Goldner and Graybeal (1965), the Bay Area Simulation Study (1968), Crecine (1968), Goldner (1968), Echenique, Crowther and Lindsay (1969), Goldner (1969a, 1969b), Wilson (1970b), Batty (1971a), Echenique, Crowther and Lindsay (1971), Goldner, Rosenthal and Meredith (1971) (***). Many other models including certain of Wilson's (****) can be seen as having originated from that of Lowry. In listing them in this way we do not wish however to overemphasise the aspect of 'continuity' and to lose sight of important innovations which certain models

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(*) In fact we have no difficulty in recognising in Lowry's model the foundation stone, and a point of departure for the whole model development mentioned here.
We may also consider the models developed by the Penn Jersey Transportation Study (Seidmann, 1964, 1969) as having originated from Lowry's model. Certain of the sectoral studies such as Herbert and Stevens' model of the housing market (1960), the industrial and service locations model and those related to the transport infrastructure (cf.: Merlin, 1968, pages 37-39). Other models developed at the same time at Lowry's though less global in their range were those of Huff (1963, 1964), Harris (1964) and Lakshmanan and Hansen (1965).
None of these models however provided anything like the impulse generated by Lowry's model, the influence of which has been described fully by Goldner (1971).

(**) Garin (1966) gave a matrix version of Lowry's model which is frequently referred to. Although Garin's version facilitated computation and hence accelerated its diffusion, it overshadowed for a longtime certain potential general developments of the model such as the non-linearity deriving from the presence of spatial constraints (which cannot be easily dealt with in a linear algebra version).

(***) For a review of the first modifications and developments of Lowry's model, see: Goldner (1971) and with reference to Great Britain, Batty (1972c).

(****) As stated by Wilson himself (Wilson, 1969a).

introduced. The applications of Lowry's model (and its variants) were numerous and of different levels of complexity. Among the first, we have those of Batty (1969a, 1969b), Echenique, Crowther and Lindsay (1969), Echenique et *al*. (1969), Batty (1970a), Cripps and Foot (1970), Echenique and Domeyko (1970), Masser (1970), Stubbs and Barber (1970), Barras et *al*. (1971), Echenique et *al*. (1973), Batty et *al*. (1974), Bertuglia and Rabino (1975), Christiansen (1975), Ayeni (1976), Ires (1976), Piasentin, Costa and Foot (1978) (*) (**).

Criticisms have inevitably been levelled at Lowry's model and with the experience of those who have applied it certain limitation and oversimplifications have been brought to light and suggestions for overcoming them proposed. The earliest modifications came from: Cripps and Foot (1969), Batty (1970b) and Broadbent (1970).

The second aspect of model development referred to (that of the formulation of alternatives to the traditional neo-classic approaches), was stimulated by the work of Wilson. The impact of his work can be attributed to his innovative vigour, the result of which can be seen in the profound influence it has on later research. We are referring here to his introduction of the entropy maximising principle. The fundamental message conveyed by Wilson was the need to steer away from the determinism and perfect rationality implied in neo-classic theory and to introduce more realistic (and even stochastic) aspects. Wilson's early work adopted this theme, Wilson (1967, 1969a, 1969b, 1969c), which was then developed further, Wilson (1970a, 1971) and Wilson (1974).

Stimulated by the work of Wilson several new directions of research were followed up, often involving the implementation of models of spatial interaction, but also sometimes purely for the sake of furthering understanding. Among the first developments orientated towards implementation were those concerned with calibration, Mackie (1971), Batty and Mackie (1972), Batty et *al*. (1973), Cesario (1973), Massey (1973), Kirby (1974), Baxter and Williams (1975), Putman (1977), Putman and Ducca (1978a, 1978b) and those involving the zoning of study areas, Broadbent

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(*) Certain of the modifications and applications cited tend towards a dynamic version and are therefore dealt with later on.

(**) Alongside the models which were directly or indirectly inspired by Lowry there are others which are often forgotten. We wish here to mention at least one of these, the statistical/econometric approach exemplified by the POLYMETRIC model (Traffic Research Corporation, 1964) and EMPIRIC (Hill, 1965, Hill, Brand and Hansen, 1966). Both of these models are described more fully in Merlin (1968, page 39).

(1969a, 1969b, 1970), Batty (1973), Batty et *al*. (1973), Batty and Masser (1975), Beardwood and Kirby (1975), Masser, Batey and Brown (1975), Openshaw (1977, 1978). A work which represented an important advance and systematic re-organisation in this respect is that of Batty (1976). Among the first contributions to the better understanding of the models were those concerning the interpretation of spatial interaction models in terms of mathematical programming; Wilson and Senior (1974), Nijkamp (1975a), Brotchie, Lesse and Roy (1979), those involving the extension of the principle of entropy maximisation in new sectors; Macgill (1977a), Macgill and Wilson (1979) and those which reconciled classical urban economics and models of spatial interation; Anas (1978b, 1979, 1982).

After the introduction of Wilson's principle of entropy maximisation another important development was that of the various economic interpretations of spatial interaction models. These interpretations by and large can be divided into two groups:

a. macro-economic approaches. In this group we have the models derived from the maximisation of consumer surplus and those based on cost efficiency;
b. micro-economic approaches. This group consists of the family of models based on random utility theory.

The principle of maximisation of consumer surplus was introduced by Neuburger (1971), Cochrane (1975) and developed principally in Coelho and Wilson (1976), Coelho and Williams (1978) and in Coelho (1979).

Another approach which is very similar to the maximisation of consumer surplus is the maximisation of accessibility developed by Leonardi (1978).

The principle of cost efficiency was developed by Smith (1978a, 1978b, 1983).

Random utility theory, which is perhaps the most important attempt to give spatial interaction models an economic base is, both because of the number of models produced and the range of possible applications, that most closely comparable to the entropy approach of Wilson. This theory had its origins in the work of Thurstone (1927) and Luce (1959) and was extended to transport and the urban context in general in the work of McFadden (1973), Manski (1973), Ben-Akiva (1974), McFadden (1974), Domencich and McFadden (1975), Lerman (1975), Manski (1975), McFadden (1976), Manski (1977), Manski and Lerman (1977), Brotchie (1978), McFadden (1978), Ben-Akiva and Lerman (1979), Brotchie (1979), Daganzo (1979), Lerman and Manski (1979), Manski and McFadden (1979), de

Palma and Ben-Akiva (1981), Leonardi (1981b, 1982a, 1982b), Smith (1982) (*).

Despite the differences, sometimes considerable, which exist in their theoretical base, the approaches derived from the theories described above can be considered equivalent to each other and equivalent also to those derived from the entropy principle maximisation. In fact, all of the approaches discussed so far, including the maximisation of entropy, lead, under fairly general hypotheses, to choice models and spatial interaction models in the form known as the logit model.

It follows that the logit model is consistent with purely aggregated non-economic hypotheses such as the maximisation of entropy in its statistical mechanics interpretation (Wilson, 1970a, Leonardi, 1977) (**), whith macro-economic hypotheses, such as the principle of maximisation of consumer surplus and cost efficiency and also with micro-economic hypotheses such as those underlying random utility theory. This equivalence has been proved by various authors, among whom are Coelho and Wilson (1977), Williams (1977), van Lierop and Nijkamp (1979) and Coelho (1983).

Many of those involved in location and transport studies have made attempts to "relax" one of the restrictive assumptions of the neo-classic approach, i.e. equilibrium wich is shared in fact by many spatial interaction models (***).

There have been two different ways of tackling the problem of transformation.

In the first, the dynamics of interacting phenomena are considered in linear terms (in other words, the variation in one quantity is seen as a linear function of other quantities).

(*) The assiomatic theory of choice (Smith, 1975a, 1975b, 1976a, 1976b) was proposed alongside random utility theory. For the relationship between the two see Williams and Wilson (1980), who trace them back to their origins: Thurstone (1927) in the case of random utility theory and Luce (1959) for the assiomatic theory of choice.
There is also another version of random utility theory which was traced by Williams and Wilson (1980) back to Quandt (1968), Niedercorn and Bechdolt (1969), Beckmann (1971), Beckmann and Golob (1971), Golob and Beckmann (1971) and Golob, Gustafsson and Beckmann (1973).

(**) The statistical mechanics analogy is not the only interpretation of the entropy maximising principle. Wilson himself (Wilson, 1970a) discusses the interpretation in terms of the theory of information. This latter interpretation was preferred and developed by various authors, among them Erlander (1977, 1980) and Webber (1979). Entropy is also used, without real theoretical justification, as an empirical device to introduce a realistic dispersion into location and transport models. The best example can be found in the work of Boyce et *al*. (1981a, 1981b).

(***) The static approach ignores not so much the time dimension of problems so much as the causal factors of evolution over time.

In the second, the dynamics of interacting phenomena are considered in non-linear terms [the variation in one quantity is seen as a non-linear function of other quantities (*)] . This development was stimulated by the need to analyse the endogenous mechanisms responsible for the interactions between the various actors and tensions between various processes which are fundamental features of an urban system and from which the non-linearities of the structure of the changes of state derive.

For a discussion of fundamental issues and general problems in the building of dynamic models we refer to Batty (1971b), Cordey-Hayes (1972), Wilson (1974), Nijkamp (1975b), Wilson (1976b, 1977), Williams and Wilson (1978) and Wilson and Macgill (1978). Forrester (1969) and others who applied Forrester's model made important contributions to the linear treatment of the problem of dynamics. Modifications to this model and subsequent developments were introduced by Kain (1969), Babcock (1970), Garn and Wilson (1970), Kadanoff (1971), Batty (1972a), Burdekin and Marshall (1972), the authors of the Special Issue on Urban Dynamics of ***IEEE Transactions on Systems, Man and Cybernetics*** (April 1972), Chen (ed.) (1972), Chen (1973), Mass (ed.) (1974), Schroeder III, Sweeny and Alfeld (1975), Alfeld and Graham (1976), Beumer et *al.* (1978) and others.

We can also include in this group the Dortmund model (Wegener, 1981, 1982, 1983) an the Turin model (Bertuglia et *al.*, 1980, 1982, Bertuglia, Gallino et *al.*, 1983a, 1983b, Bertuglia, Occelli et *al.*, 1983) although the treatment of the residential subsystem in the latter has more in common with the second approach (**).

In the non-linear treatment we can distinguish a number of different influences,

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(*) The relaxation of the hypothesis of equilibrium and the introduction change, gave rise to the production of models is which the time dimension appears only as a descriptive feature of a process of comparative statics (in fact comparative analysis compares different equilibrium states without considering how the transitions from one to the other occur). This was what was done initially with the Lowry model (Crecine, 1964, 1968, 1969a, 1969b, Seidman, 1969, Dickey, Leone, Schwarte, 1971, Batty, 1972a, 1972b, Sharpe et *al.*, 1974, Bertuglia and Rabino, 1975, Sharpe, Brotchie and Ahern, 1975, Ayeni, 1976 and IRES, 1976).
For an analysis of the passage from comparative statics to dynamics see: Wilson (1978a, 1978c). A more detailed discussion of the meaning and significance of the expression "dynamic" is given in Martin, Thrift and Bennett (eds.) (1978) particularly in the introduction by the editors.

(**) Important work has been done by Rogers (1971, 1975) and his school on population studies, which are those most closely associated with our present study.

on the kind of approach and subject-matter treated. Some models make use of dynamic generalisations which come directly from geography and the regional sciences, such as central place theory (Curry, 1969, White, 1977, 1978, Wilson, 1978b, Allen and Sanglier, 1979a, 1981a), or diffusion theory in its various forms (Curry, 1978, 1982, Ralston, 1983, Sonis, 1983) and the dynamic version of optimum land use (Isard and Liossatos, 1972, Domanski, 1973, Isard and Liossatos, 1975, 1979).

Others make use of recent mathematical or physico-mathematical theories which have been applied to the analysis of the dynamics of spatial phenomena. It is interesting to note in this connection that those involved in producing such models tend to be regional scientists or physicists and mathematicians interested in urban problems. A predominant role is played here by the vast group of studies inspired by the catastrophe theory which was applied at urban level by Amson (1974, 1975), Casti and Swain (1975), Amson (1977) and Clarke and Wilson (1983a, 1983b) and to the analysis of economic development and decline, Casetti (1981a, 1981b). The most important development was however the analysis of the dynamics of urban subsystems, particularly that of the service subsystem, undertaken by Wilson (1976a, 1978b, 1978c, 1979a, 1979b, 1979c, 1981a, 1981b), Poston and Wilson (1977), Harris and Wilson (1978), Wilson and Clarke (1979), Beaumont, Clarke and Wilson (1981a, 1981b), Harris, Choukroun and Wilson (1982) and also Lombardo and Rabino (1983a, 1983b) and Rijk and Vorst (1983). A group of studies derived from 'ecological' models, concerning in particular competition between species, where emphasis was placed on aspects of structural stability or instability includes the work of Dendrinos (1977, 1978, 1979, 1980a, 1980b, 1981b, 1982), Dendrinos and Mullaly (1981a, 1981b), also Day (1981) and Monaco and Rabino (1984). The theory of dissipative processes developed by the Brussels school inspired another group in an attempt to extend the applications to the analysis of urban systems. Here we refer to Allen et *al*. (1978, 1979a, 1979b, 1982), Allen and Sanglier (1978, 1979b, 1981b), Allen, Boon and Sanglier (1980) and Crosby (1983). A further group developed from the theory of synergetic processes, which was extended to the analysis of the dynamics of social and spatial interactions. We include in this group the work of authors not only from the field of synergetics but also regional scientists whose work shows a great similarity of approach. The essential difference between the kind of dynamic processes considered by this last group and the preceding one is the speed of the process — those in the former group being essentially slow processes (such as changes in housing stock or services) and those in the latter being relatively rapid (such as mobility of population). We cite here by way of example Bertuglia and Leonardi (1979), Weidlich and Haag (1980), Leonardi and Campisi (1981), Haag and Weidlich (1983), Leonardi (1983), Weidlich and Haag (1983) (*) (**).

We see in this non-linear approach, which has been adopted by an increasing number of studies how important a stimulus the application of mathematical techniques has been. We refer especially to differential topology (Chillingworth, 1976) which includes several important theories, in particular catastrophe theory (Thom, 1972) and also to the theory of dissipative processes (Nicolis and Prigogine, 1977) and the theory of synergetic processes (Haken, 1977).

A general criticism of neo-classic assumptions has been made by the urban economists who have proposed alternative economic paradigms for urban analyses in general and also for transport-location analyses. The most prodigious of these seem to be the neo-Marxian (or neo-Ricardian) paradigms deriving from the essentially non spatial theories of Sraffa(1960), Garegnani (1970), Spaventa (1970), Morishima (1973), Pasinetti (1974, 1977), Steedman (1977), Pasinetti (1981), Steadman and Sweezy (eds.) (1981) and recently, covering spatial aspects, Scott (1976, 1979, 1980, 1982) and Sheppard (1981, 1983a, 1983b).

There is also another group of neo-classic non Walrasian approaches which includes the static models of Drèze (1975) and Benassy (1975), the dynamic models of Varian (1975), Kornai and Weibull (1978) and Weibull (1983), but these do not as yet introduce of the spatial dimension explicitly.

As well as the evolution of scientific thought and the connected production of explanatory theories which we have attempted to describe in the preceding pages there has also been a notable growth of normative techniques involving optimisation and testing.

We have first of all the extremely important contribution of Operational

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(*) From the point of view of the treatment of the residential subsystem we can include in this group the Turin model (Bertuglia et *al.*, 1980, 1982, Bertuglia, Gallino et *al.*, 1983a, 1983b, Bertuglia, Occelli et *al.*, 1983), and the Dortmund model (Wegener, 1981, 1982, 1983).

(**) Population studies which can be included in this group: Rees and Wilson (1977), Ledent (1978), Okabe (1979), Sikdar and Karmeshu (1982) and Sheppard (1983c).

Research which developed: (i) models based exclusively on transport costs (*) (cf.: Eilon, Watson – Gandy and Christofides, 1971, Handler and Mirchandani, 1979, Halpner and Maimon, 1982, Coelho, 1983, Hansen and Thisse, 1983); (ii) models based on plant costs with increasing returns (cf.: ReVelle, Marks and Liebman, 1970, Francis and Goldstein, 1974, Bartezzaghi, 1979, Coelho, 1983), models with technological constraints (cf.: Salkin, 1975, ReVelle, Cohon and Shobrys, 1981, Coelho, 1983) and (iii) models taking into account non-perfect rationality on the part of the decision-maker (cf.: Leonardi, 1978, Leonardi 1981a, Palermo, 1981, Wilson et *al.* 1981, ***Sistemi Urbani, 3***, 3, 1981, Coelho, 1983) and models with multiple objectives (Haimes, 1977, Nijkamp, 1977, Nijkamp and Spronk, 1981, ReVelle, Cohon and Shobrys, 1981, ***Sistemi Urbani, 3***, 3, 1981).

There has also been a growth of methods designed as aids in decision-making and evaluation in location processes. There is not space here to describe these in full but certain aspects deserve to be mentioned – in particular those methods which attempt to link efficiency and optimisation with the satisfaction of decision-makers (Simon, 1955). These include Goal programming (Charnes and Cooper, 1961) and more recently vector optimisation (Geoffrion, 1968, Zeleny, 1974), methods based on non-dominated structures (Yu, 1973a) and methods based on outranking and the new assiomatics of Roy (1973, 1974, 1975, 1976, 1977, 1979a, 1979b). Although it is based on satisfaction, Goal programmin can be considered more a modification of rational strategy than a pluralistic approach and in addition does not seem to take the behaviour of decision-makers fully into account (cf.: Ostanello, 1980). The interactive methods are moving explicitly in this new direction and are part of the so-called "hybrid approach" with the decision-makers' behaviour being inserted in the model (Aubin and Naslund, 1972, Geoffrion, Dyer and Feinberg, 1972, Steuer, 1977, Nijkamp and Spronk, 1979). A common strategy of these methods is that of presenting the decision-maker with a succession of new alternatives, asking him to express his preferences. They place emphasis on the decision-making process rather than the decision itself (**). The model is seen as a method of support to the

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(*) In order to keep the number of bibliographical references to a minimum in this case and in those immediately following we refer to review works.

(**) The convergence on a solution is facilitated in many of these methods by having a 'point of reference' eg. the "perfect solution" of Geoffrion and Dyer (Geoffrion, Dyer and Feinberg, 1972), the "utopia point" of Yu (1973b), the "target" of Roy (1975), the "ideal" of Zeleny (1976). This point of reference may be redefined with the interactive process eg. the "evolutive target" of Roy (1975), and the "dispeaced ideal" of Zeleny (1976).

process of solving the decision-maker's problem.

1.3. *The most promising aspects of current research*

From the above survey, the three aspects of current research which emerge as most productive and promising for the future are:

a. from the behavioural point of view, the progress from deterministic models to stochastic models;

b. from the point of view of the spatial and time structure the development of dynamic models from static ones;

c. from the point of view of economic theory the contrast between neo-classic theory and the new urban economics and neo-Marxian (or neo-Ricardian) theory.

These three aspects if followed up are likely to bring changes both in the way of analysing the various phenomena connected with location and transport interrelations and in the approaches traditionally used for solving the inherent theoretical and methodological problems. We shall in the following section look at these phenomena according to the classification introduced in 1.1., attempting not only to describe the state-of-the-art but also to assess the possible impact of new development. In addition we have selected from the vast range of theoretical and methodological problems, three which can be considered "key" points in the understanding of static and dynamic behaviour of systems of location and transport:

a. spatial choice behaviour models;

b. mechanisms for the formation and spatial differentiation of prices;

c. the technological structure of intersectoral transactions and mechanisms of production and consumption.

2. Interrelations between location and transport in human settlements

2.1. *Location of economic activities and commodity flows*

2.1.1. Introduction

While the study of the relationships between location and flows of people (especially for residential and commercial activities) can be considered relatively developed and with recognisable unifying elements among the different approaches (see sections 2.2. and 2.3.), the interrelations between economic activities and commodity flows has received relatively scarce attention. The work which has been done appears not to have a common thread and is rather difficult to fit into a general scheme.

There are differences in the ways the various disciplines have approached the problem and in the depth to which they have explored the subject. Without doubt those who have done more than anyone else are the economists, though often they have ignored the question of space, or introduced it at a highly aggregated level (eg. usually at regional level). Geographers and those regional scientists orientated towards physical planning have tended to introduce a more refined spatial disaggregation but at the cost of an exogenous treatment of the structure of intersectoral flows, considering them as given and not explaining them.

In order to make a systematic survey of the possibility of building a general theory of location-commodity flows relationships, and also to identify directions for future research, it will be useful to introduce a broad classification of problems, based on some qualitative differences in the type of commodity flows to be considered.

We distinguish first of all between flows of goods towards consumption and flows of goods towards production.

The former can be further broken down into flows of a single good and flows of multiple goods. For the latter no subdivision is necessary.

There is however a third aspect, distinct from but closely linked to, this basic subdivision – the relationship between flows of goods, the location of economic activities and the labour market. An integrated analysis of the connections between these three phenomena is fundamental to the understanding of the structure and dynamics of a system of settlements. Economic activities receive as inputs not only products from other sectors but also labour from the working populations. In addition, as they sell finished products to consumers, they condition and are conditioned by residential location.

A comprehensive model of the inter-relations described above does not exist, but given their complexity this in some ways is understandable. It is surprising however that two modelling traditions such as inter-regional flows of goods and urban models of the Lowry type have practically never been brought together. Their integration could be extremely useful for future developments, a theme which we shall take up again later.

2.1.2. Model of flows of a single commodity towards the consumption

The models of which we shall speak briefly here go back to the formulation proposed by Samuelson (1952) and are discussed in another chapter of this book by Beckman. The basic Samuelson model has been subject to various developments and critical revisions (Takayama and Judge, 1964, Sheppard and Curry, 1982). It is based on two main assumptions:

I) the existence at each point (or in each zone) in space of a net demand function (local consumption minus local production) which depends exclusively on local prices;

II) the embedding of market equilibrium into an optimisation problem, in which the total benefit (consumer surplus plus producer's surplus) is maximised.

While the second assumption is less open to criticism, at least from a neo-classic viewpoint, the first needs to be looked at more closely.

First of all we should note that these models, even though they have been classified as involving flows "towards consumers", deal in fact with an aggregation of all functions of production and final consumption in a single demand function. The net demand in each zone is not necessarily the final consumption, but includes in a single expression all the intermediate and final consumptions in that zone, both for the productive system and the consumers. The technological structure of production is therefore ignored, as assumptions on intersectoral transactions (for a single product) do not appear explicitly. The level of consumption in each zone is determined exclusively by the local price. This assumption seems to be reasonable as far as consumption is concerned, if the delivery costs are paid by the producer. In this case the consumer effectively pays the local consumer price, which becomes the determinant factor for the consumption level. It must not be forgotten however that the net demand is defined as the difference between consumption and local production and even if the hypothesis of dependence on local price is acceptable for consumption it will not be so for local production. In such systems the production is, by definition, orientated towards export to other zones. In addition it is the producer who pays the delivery costs of commodities.

It therefore seems reasonable, though, contrary to what is accepted in the classic model, to assume that local production is a function not of local prices but all prices in all zones, plus the relative transport costs.

The only logical way of accepting a supply function which depends only on local price seems to be by reversing the classic assumption, (that transport and delivery costs are paid by the exporter) and taking them to be paid by the importer, i.e. the consumer. In this case it is the exporter who fixes the local price, and this plus the transport costs in the zone of consumption must be paid by the consumer at the final destination.

At this point another contradiction emerges. If the supply can be a function of local price only, it is not so for the demand. In fact if the consumer must buy the product in the various zones of production paying the respective local prices plus transport costs, the demand in each zone is a function of all the prices in all the zones as well as transport costs and not only the local price.

An intrinsic contradiction therefore seems implicit in the very concept of net demand and the way in which it is represented in the classic model.

This underlines the fact that it is necessary to introduce an explicit representation of the productive structure and producer behaviour and that the latter cannot be aggregated with that of the consumer without losing information fundamental to the understanding of the system.

2.1.3. Models of flows towards consumers for multiple goods

The same criticisms which have been levelled at the case of single commodity flows naturally hold also in the case of multiple commodity flows and need not be repeated.

However, accepting for a moment the idea of a function of net demand depending only on local price, there can be a number of different models, according to the different assumptions made about the interactions between the different products either at the time of consumption or production.

From the point of view of the consumption, which is our main interest here, there are two distinct cases according to whether we assume the net demand for each product is exclusively a function of local price for the product or a function of all local prices for all products.

In the first case, in the absence of other constraints, the problem of multiple products can simply to reduced to several independent problems involving a single product.

In the second case we have a system of demand equations in which all products have interdependent consumptions and the problem becomes more complex. While systems of demand equations for bundles of multiple products have already been proposed and widely studied in economic literature, their combination with the spatial dimension through a structure of interzonal imports and exports has received little attention. These models need therefore to be further developed if they are to be useful in this context.

The existence of multiple products underlines even more the need to distinguish clearly between final consumption and intermediate consumption. The latter, if formulate for more than one product, is the same as the disaggregated specification of functions of production for each sector, and is connected with the analysis of intersectoral interdependeces, which is generally treated as being a separate problem from that of inter-regional commercial flows.

This theme is discussed in section 2.1.4..

2.1.4. Flows of goods towards production and sectoral interdependencies

The analysis of intersectoral transactions is dominated by the input-output approach which postulates the existence of a constant matrix of transaction coefficients or, in other words, linear technology.

The subject is already too well known to require a detailed description here.

We limit ourselves to a brief mention of the attempts to introduce multi-zonal spatial disaggregation into the input-output approach, thus making the relationships between the structure of the production system, the location of production units and the interzonal flows of goods explicit.

The first and simplest way in which spatial disaggregation can be introduced is that of extending the concept of constant intersectoral coefficients to that of constant intersectoral-interzonal coefficients. This purely descriptive approach is the oldest and is explained in Isard (1960).

A more recent and more explanatory approach involves the combination of the input-output structure with a model of spatial interaction of the type commonly used in the analysis of movements of people. An example is the work of Macgill and Wilson (1979), in which spatial disaggregation is obtained by applying the maximisation of entropy method and using sectoral interdependence as constraints. A similar approach is proposed in Bertuglia and Leonardi (1980) and in Batty (1983). The same kind of spatial disaggregation, by means of logit-type models is used by Sheppard in this book, even though in this case entropy maximisation is not invoked as a justification.

What seems to be missing is a model in which the linear technology of input-output models is effectively combined with an economic model of interzonal flows of the Samuelson type.

Even though the construction of such a model must obviously be left to future research, we suggest here a possible structure. Accepting the two paradigms of linear technology and the maximisation of total surplus, an integrated model could be devised incorporating the following elements:

a. a function of final demand (or consumption) should be defined for each zone. This substitutes the concept of net demand, the contradictions of which have already been discussed;

b. the linear technology is imposed as a constraint on total intersectoral transactions. This constraint involves both production levels and final demand;

c. spatial disaggregation, both for intersectoral transactions and delivery to final demand, is obtained by subtracting total transport costs from the total benefit function as in the classic Samuelson model;

d. a dispersion term (eg. entropy) should be added to total benefit. In this way the function to be maximised contains a term of net surplus dependent on the function of final demand, a term of transport costs dependent on intersectoral-interzonal transactions and delivery flows to final demand, and a term of dispersion dependent also on intersectoral-interzonal export flows.

In such a model the equilibrium state is obtained by maximising total surplus, subject to the constraints specified in b. above.

We can predict certain salient features of the structure of the solution of this conjectured model.

Above all the dual solution (shadow prices associated with the constraint b.) would provide a mechanism of price formation in which the relations between consumer prices and intersectoral transactions would appear. This, to a certain extent, would combine a mechanism of price propagation of the type described by Sraffa (1960), Morishima (1976) and Sheppard (in his chapter in this book) with the classic concept of demand. The fundamental difference between the Sraffa-type models and the kind of solution proposed here is that while in the former even consumption is treated through linear-technology in our case it is described by a demand function, which is in general non-linear.

Secondly, the presence of the dispersion term of interzonal flows would produce models of a similar structure to classic spatial interaction models (eg. logit models), containing however prices as well as transport costs. This, apart from the theoretical improvement, is a great advantage when it comes to application, because of the relative simplicity of the calculations required by such models.

Finally, from the point of view of the understanding of the spatial structure of a multi-sector economy, the proposed model would be a definite advance on the purely physical models of the Lowry type. Besides information on location and spatial interaction, it would also provide information on prices, on their formation and spatial differentiation. In addition, a factor which should not be underestimated, it would provide all this through a precise economic interpretation, something which was lacking in the Lowry type models (with the exception of certain recent developments proposed by Anas, 1983).

2.1.5. Commodity flows, location of economic activities and the labour market

The model outlined in 2.1.4., even though it promises to overcome various gaps and contradictions in existing models, is still deficient in one fundamental aspect.

The resident population is seen solely as consumers and not as a labour force and part of the productive process. It is important that this relationship between population and production should be taken into account since:

a. for the production, manpower is an input as much as any other factor of production. It also constitutes the main feedback in the wage — profit cycle. Wages paid to workers determine final consumption and hence production;

b. for the resident population the wage deriving from labour in a productive activity determines the demand for all goods and services including those associated with his ability to settle in a given zone, hence affecting housing and transport.

Naturally, the role of labour is contained in both traditional input-output models and their neo-Marxian versions and in the Lowry model and its extensions. However, in all the cases mentioned here labour is treated as a linear function in the same way as any other productive sector. Prices and wages in these models do not play a fundamental role in determining levels of employment and consumption (they are in fact either ignored or assumed to be exogenous).

Here we wish to put forward a generalisation of the model proposed in 2.1.4., which integrates labour and endogenously generates wages and prices.

Basically such a generalisation is simple. In the same way that final demand functions and prices were introduced previously, it is possible to introduce wages and demand functions for labour for each productive sector. Naturally, the demand functions for final consumption will have to be reformulated in order to take account of the constraint on consumption set by disposable income i.e. wages.

The total surplus function would contain two additional terms — the producer's surplus, related to the demand for labour, and the cost of transport, associated with journeys to work, paid by the population. Both the labour input and final demand would appear explicitly in the constraints on intersectoral transactions.

It is important to note that the system described above would be essentially governed by production rather than by consumption. Consumers partly control prices through the final demand functions, but are constrained by wages. Producers control wages through the labour demand functions but are not directly constrained by consumer prices. In fact, through the constraints on intersectoral transactions, producers control to a considerable extent both wages and prices.

The solution of this modified version of the model would have two additional advantages.

Firstly its dual solution would provide a joint mechanism not only for prices, but for price and wage formation. Secondly a model relating journeys to work, transport costs and wages would be generated, probably with a structure similar to a classic spatial interaction model.

In conclusion, the model obtained should provide all the information of a classic Lowry model plus commodity flows, prices and wages.

2.2. *Location of services and journeys generated by their use*

In this section we intend to outline the development of theory relating to location of services and to identify the most important stages in this development.

The starting point from which location theory and especially service location theory grew is the neo-classic approach (Beckmann, 1968). This approach forms the nucleus of the economic activity equilibrium theory (see Beckmann's chapter in this book). The basic characteristic of this theory is the achievement of an equilibrium state where "firms" (here the suppliers of services) maximise profits and users of services maximise utility.

We can see here in the concept of optimal location the influence of Hotelling (1929). According to him the location of a service was optimal when the costs of its use were in equilibrium.

The principal limit of the economic activity equilibrium theory is that it is necessarily founded on the idea of equilibrium. This is a condition rarely achieved in reality and even if it occurs, it cannot be said to be always essential.

Other disadvantages are the impossibility of dealing with indivisibilities, externalities and imperfect rationality of decisions makers.

Service location in fact involves the indivisibilities such as fixed costs of provision or capacity constraints of facilities. These factors lead to combinatory allocation problems, i.e., how to locate m service facilities choosing between w ($w > m$) possible locations. The equilibrium approach cannot resolve such problems unless we assume homogeneity of space. In this case optimum location can be found using "marginal" analysis.

As we have said above, externalities are another element that the economic activity equilibrium theory cannot deal with. By externalities here we mean such factors as spill-over, diffusion, economies of agglomeration and environmental effects. It must be recognised however that the introduction of externalities would not compromise the fundamental assumption of the theory, the equilibrium state, but would make this equilibrium non-optimal.

A further limitation of this approach is the assumption that decision-makers are rational. It assumes for example that users of services seek to maximise their expected utility and that suppliers of services seek to maximise their profits. In addition it is supposed that all the decision-makers have the same tastes and preferences, which is clearly unrealistic in most cases.

To conclude this rather brief description we should add that the economic activity equilibrium theory is of course static. It can be used to describe equilibrium situations and for comparative analysis of such situations but cannot explain how equilibrium is reached.

Despite these limitations, the theory has nevertheless been a reference point for a number of developments in location theory which have succeded in overcoming the problems mentioned and opened new fields of research. These we now describe in chronological order.

We begin with the observation that numerous methods of operational research still in use for service-location make use of the approach described above. These methods are characterised by the type of objective function employed which is typical of the neo-classic approach and is based on the concept of "efficiency" (minsum). According to this concept the function to be minimised is some measure of disutility for the whole system eg. total travel cost. Objective functions of another type came from the consideration of the "worst case" (minmax). Here the function to be minimised is a measure of disutility, eg. the travel cost of the user in the least favourable conditions. More recently a new formulation of the objective function was based on the concept of "equity", eg. the redistribution of profits or income.

Following Colorni (chapter in this book) we list here a summary of other models which have been developed from these origins, classified according to type of location factor:

a. location based on transport costs (a review of these can be found in Eilon, Watson-Gandy and Christofides, 1971 and Coelho, 1983).

   Further assumptions are that there is:

   1. a single indicator of preference, based on costs;

   2. perfect rationality of decision-makers;

3. no technological constraints (eg. minimum or maximum capacity);
4. no plant costs.

These, it can be seen, are typically neoclassic models. They have been used both in continuous space (Hansen and Thisse, 1983) and discrete space (among others, Handler and Mirchandani, 1979);

b. location based on transport and plant costs (a review can be found in Francis and Goldstein, 1974 and Coelho, 1983).

The following assumptions still hold:
1. a single indicator of preference based on costs;
2. perfect rationality of decision-makers;
3. no technological constraints.

This kind of model is no longer strictly in the neo-classic mould. The presence of plant costs generates indivisibilities which the economic activity equilibrium theory cannot deal with for the reasons explained above;

c. location with technological constraints (for a review of these models see ReVelle, Cohon and Shobrys, 1981, and Coelho, 1983).

The following assumptions remain valid:
1. a single indicator of preference, based on costs;
2. perfect rationality of decision-makers.

These models resolve the so-called location-allocation problems;

d. location with non-perfectly rational decision-makers (Wilson et *al.*, 1981, Leonardi, 1981a).

The following assumption remains valid:
1. a single indicator of preference, based on costs.

In order to introduce non-perfect rationality of decision-makers and differentiation of their tastes and preferences, a random component is introduced in the utility function of decision-makers. Another interesting and more recent development is the introduction of a random component not only in decision-makers' behaviour but also in the transport network. In this way we can take into account stochastic aspects of the network which are analysed using probability graphs (Berman and Odoni, 1982). Another way of introducing a certain dispersion into decision-makers behaviour is through the entropy models (Wilson, 1974).

These models, other than including dispersion, could almost be considered multiple objective models. Their objective function consists of minimisation of costs (as first objective) and maximisation of decision-makers' surplus (as second objective) or maximisation of accessibility, which is analogous (Coelho, 1983);

e. location with multiple objectives (Nijkamp and Spronk, 1981, ReVelle, Cohon and Shobrys, 1981).
In these models we have an objective function with more then one objective and these objectives may even be conflicting. They can be quantitative or qualitative. Very often the function in fact consists of two objectives, one quantitative (minimisation of total travel costs eg. based on the efficiency criterion) and one qualitative (improvement of the service quality, based on the worst case criterion).

Together with these models we can consider the multi-criteria models. These, as they are able to deal with qualitative and quantitative information simultaneously, are useful for the determination of planning policy(see Voogd's chapter in this book).

In addition to the developments outlined above there have been others which have originated from them:

1. the use of methods of random search and global optimisation to solve combinatory problems such as those mentioned in b. (Camerini, Colorni and Maffioli, 1983);

2. the subdivision of the decision-making structure into levels.
Here the phenomena of competition between decision-makers are considered and games theory also used.
This development derives in particular from the models with multiple objectives referred to in e. above;

3. disaggregation of the model variables.
This disaggregation (see Wilson's chapter in this book) is achieved according to:
a. type of good or service;
b. type of structure for the provision of goods or services;
c. type of user;
d. mode of transport.
Of these the most interesting and most recent is the second, the type of structure for the provision of goods or services. By taking into account differences in structure it makes it possible to analyse different and competitive location behaviour.
A further disaggregation is recognisable in the costs borne by the suppliers of goods or services. They can be split down into fixed costs of provision and

running costs.

The introduction of disaggregation makes the models far more complex both from the computational and data collection point of view. To reduce this disadvantage, at least in part, techniques of micro-simulation can be used (Clarke and Spowage, 1982).

A final observation here is that the higher the level of disaggregation of the model, the more numerous are likely to be the phenomena of bifurcation of which we shall speak in the next paragraph. This arises from the non-linearity and large number of interdependencies present in hyghly disaggregate models;

4. dynamics of building stock.

Despite its acknowledged importance (Wilson in this book) the analysis of stock dynamics is still relatively under-developed. The need for an analysis of this kind derives from the fact that service facilities are located in an already structured environment in which the major problem is how to deal with existing stock – whether to expand, demolish or relocate.

The dynamics of the stock can in its turn cause bifurcation and instability (Wilson, 1981a). These phenomena appear when small changes in a parameter near a critical point determine large structural changes. Only dynamic analysis is able to take account of this type of phenomenon. For the planner the existence of bifurcation and instabilities has the following implications:

a. negative: if the system is about to reach an undesired state, a modification of the parameters is required to prevent this;

b. positive: it is possible to guide the system to a desired state with small adjustments (eg. limited investments). These are sufficient to bring the parameters to the critical value at which the desired state of the system is reached;

5. multi-level systems (Leonardi, 1981a, Leonardi and Tadei, 1981, Bertuglia, Leonardi and Tadei, 1983).

The consideration of multi-level service systems is made necessary when for example the hypothesis of the single-purpose trip no longer holds. The single-purpose trip (home → service → home) is frequently and more realistically replaced by the multiple-purpose trip (home → service 1 → service 2. . . → service n → home). The analysis of multi-level services is also necessary for studying the interrelations which exist between different kinds of services and the effects that different organisational and functional scenarios have on the overall spatial configuration. Multi-level services are usually studied using "nested logit" models

(McFadden, 1978).

Having examined the work done up to the present in the field of service location we shall now suggest two features which could possibly be included in a programme of future research:

a. the introduction of dynamic models of information diffusion (de Palma and Lefèvre, chapter in this book) into the analysis of services, with the aim of modelling the spatial behaviour of service demand and its dynamics. In this case too bifurcations and instabilities worth analysing may arise;

b. the use of the cost efficiency theory (Smith, chapter in this book) which is applied to the study of congestion phenomena in a transport network. However it could easily be extended to deal with a different kinds of congestion, like for example the congestion of services or housing markets.
   Probably the generalisation of the classic spatial price equilibrium models with the introduction of congestion effects in the network of commodity flows, proposed in this book by Smith, could be extended to the problem of service location by considering congestion as well as travel and use of services as a price.

2.3. *Residential location and journeys to work*

We present here a brief review of the development of residential location theory and identify the most important contributions to the theory made in this book. In the final part we describe a number of possible future developments.

The most important milestones in the development of residential location theory are the models of Alonso (1964a) and Muth (1969), which are an extension of the work of von Thünen (1826) to the urban context. Alonso and Muth deal with the problem of urban land-use in which residential development obviously plays an important part.

They describe the spatial equilibrium of the city a competitive market for urban land. In addition, in their definition of spatial structure of residential areas, special emphasis is given to the effects of trade-off between accessibility and space.

These models are static and, at equilibrium, produce a spatial structure in which both residential density and land rents decrease monotonically outwards from the city centre. Land development is dense (there is no vacant land) and residential areas are structured in concentric characterised by different kinds of housing.

Alongside the models of Alonso and Muth (which are so similar that we shall from now on refer to the Alonso and Muth model) is the model of Herbert and Stevens (1960). This, unlike the Alonso and Muth model, is not theoretical but an operational model, determining residential location through linear programming. It is an extension of Alonso's theory to a policentric market in which households, divided into numerous groups with different tastes and preferences, choose their residential location. Herbert and Stevens' model was taken up and made even more operational by Harris, Nathanson and Rosenburg (1966) and later reinterpreted by Wheaton and Harris (1970).

Other contributions to residential location theory have been made by Wingo (1961), Wheaton (1972) and Mills (1967, 1972) among others. Alongside these we must not forget the more operational models such as those of Lowry (1964), the TOMM model (Time Oriented Metropolitan Model) (Crecine, 1964, 1969b), the BASS model (Bay Area Simulation Study) (Goldner and Graybeal, 1965, Bay Area Simulation Study, 1968), and the PLUM model (Projected Land Use Model) (Goldner, 1968, Goldner, Rosenthal and Meredith, 1971).

What was lacking until the beginning of the seventies was a model capable of bringing together the two different modelling approaches then emerging — the theoretical one and the descriptive operational one. Only the NBER model (National Bureau of Economic Research) (Kain, Ingram, Ginn, 1972) managed to achieve such an integration. This model which has a detailed breakdown both of the demand (households) and the supply (building stock) has been for many years the most important point of reference for the analysis of residential location.

Before we look at the more recent models it would be useful to recall the theoretical foundations of the models of Alonso, Muth, Herbert and Stevens. All of these models are based on the theory of micro-economic behaviour. They assume that:

a. there are different kinds of household, each of which is homogeneous in relation to the utility functions;

b. each household has a certain level of expected utility and is willing to pay for housing in the various zones a price consistent with its expected utility;

c. the competition between different kinds of households modifies the levels of expected utility so that all the households are allocated.

From this it is clear these models they assume the housing market to be in a state of perfect competition, which is of course not realistic. In fact we find situations of disequilibrium in which not all households find their optimal residential location — some remaining below their level of expected utility, others exceeding it. In order to take this into account Anas (1973) constructed a model of dynamic disequilibrium. This model gives a measure of disequilibrium expressed as a deviation from the global optimal solution of Herbert and Stevens model. Anas in fact reinterpreted their model in terms of entropy maximisation.

This is similar to what was done by Senior and Wilson (1974) as we shall discuss later. What characterises Anas' model and differentiates it from the work mentioned here is his interpretation of the residential choice behaviour of households, justifying it in micro-economic terms and not in terms of entropy. In addition this model is the only one mentioned so far which is not static.

It is interesting to see how, using Alonso's theory as a base, other dynamic models of residential location have evolved.

Some of the most important work has been done by Fujita (1976b), whose first contribution was to give a dynamic version of Alonso and Muth's and Herbert and Stevens' models. In the dynamic version of Alonso and Muth's model the working of the land-use market was described, and in the dynamic version of Herbert and Stevens' model this working became normative in order to reach an optimal situation. Fujita showed that it is possible to develop a unifying theory capable of including both descriptive and normative aspects of the dynamics of urban land-use.

Senior and Wilson (1974) introduced entropy to what had until then been a purely neo-classic approach. They use the principle of entropy maximisation to assign a set of households to a set of residences, both sets being broken down into categories. It can be argued that Herbert and Stevens' model is a special case of Senior and Wilson's disaggregate spatial interaction model.

A similar approach comes from Los (1978). He too uses entropy models, but the principal innovation is the use of endogenous prices.

The models of residential location described up to now, with the exception of the Bay Area Simulation Study (1968), determine the allocation of households to a housing stock which is given and fixed. In other words, we have a dynamic demand but not a dynamic supply. We shall see next how the problem of treating the dynamics of demand and supply jointly has been tackled in recent years and how it could be the subject of useful future research.

Another and new approach to residential location analysis came from the introduction of the concept of scarcity (Kornai and Weibull, 1978, Kornai, 1980). This approach grew out of the study of planned economic systems and focussed on the analysis of a housing market in conditions of chronic shortage, making use of models based on queue theory.

A dynamic model of the housing market taking into account a dynamic stock as well as a dynamic demand was developed by Snickars (1978). His model, which was certainly inspired by Kornai and Weibull's work, is deterministic and has exogenously fixed prices.

Weibull's contribution to this book is a natural development of this work. He develops a dynamic model of demand and stock which describes a market regulated not exclusively by prices but also by other economic signals such as the negative externalities due to scarcity (eg. access time to the various housing markets). The prices are endogenous and a dynamic process of demand-supply interactions is hypothesised in conditions of disequilibrium.

Finally, a last approach to the problem of residential location, developed in parallel to the entropy approach, consists of the analysis of behaviour at micro-level using random utility theory (Lerman, 1975, 1979, McFadden, 1978, de Palma and Ben-Akiva, 1981). The most well-known product of this approach is the multinomial logit model used to describe choice behaviour of a user faced with a set of alternatives. An interesting application was made by de Palma and Ben-Akiva (1981). They construct a dynamic model of residential choice in which the transition rates are given by logit models. One limitation of this model however is that the attraction factors, used in the formation of transition rates are assumed to be exogenous and not time dependent. This has been overcome by Leonardi (see his chapter in this book), who makes the factors involved in the evaluation of alternatives on the demand side endogenous (as well as prices) and gives a dynamic version of the evaluation process based on future expectations. In addition he constructs a joint model of residential and labour mobility which produces, even in conditions of equilibrium, a flow structure rather different from the gravitational type.

As previously said, relatively scarce attention has been paid until now to the interactions between stock and demand dynamics. This could well be a fruitful theme for future research, in particular if a way were found of introducing dynamic stock and dynamic demand jointly within the framework of random utility theory. More specifically, it would be interesting to see if Wilson's dynamic equations for housing stock and Leonardi's dynamic equations for demand, both proposed in this book, could be combined.

2.4. *Location and transport in the urban system*

2.4.1. Introduction

As indicated in the heading to this section we are mainly interested here in the city as a system. The relationship between location and transport in the city (considered as a whole without stressing its internal structure, is dealt with) in the next section 2.5..

Most of what has already been said about transport and location of industry, housing and services pertains also to the urban system, as these subsystems are obviously fundamental components of the urban system and what holds for each of them individually in general holds for the whole system.

To analyse to what extent our previous conclusions are applicable to the urban system as a whole we shall show briefly how an urban model can be expressed in terms of mathematical programming (Macgill and Wilson, 1979).

Much of the discussion in this book in fact refers to the possibility of formulating different equivalent mathematical programming versions for spatial interaction models.

The well-known Lowry model will be taken as a reference-model and, from among the various different programming versions, the maximisation of consumer's surplus has been chosen.

First of all we must formulate the function to be optimised for the joint processes of residential and service location

$$\max_{\{T_{ij},\, S_{ij}\}} Z = -\frac{1}{\beta_1}\sum_{ij} T_{ij} \ln T_{ij} + \sum_{ij} T_{ij}\left(\frac{\alpha_1}{\beta_1}\ln W_j^{res} - c_{ij}\right) - \frac{1}{\beta_2}\sum_{ji} S_{ji}\ln S_{ji} + \sum_{ji} S_{ji}\left(\frac{\alpha_2}{\beta_2}\ln W_i^{ser} - c_{ji}\right), \qquad (1)$$

where:

T_{ij} and S_{ji} are, respectively, the number of journeys to work from i to j and journeys to services from i to j;

W_j^{res} and W_j^{ser} are, respectively the residential and service attractiveness factors;

c_{ij} are generalised transport costs;

$\alpha_1, \alpha_2, \beta_1, \beta_2$ are Lagrange multipliers.

Then we must write the constraint equations of the two location processes, taking into account their reciprocal interdependence (according to Lowry):

$$\sum_i T_{ij} - \gamma_{1j} \sum_i S_{ji} = 0$$
$$\sum_i T_{ij} - \gamma_2 \sum_j S_{ji} = E_i, \qquad (2)$$

where:

E_i are jobs in the base sector;

γ_{1j} and γ_2 are parameters defined according to the urban economic base theory.

Equation (1) and (2) provide the mathematical programming version of the Lowry model we are looking for. To this result we can apply the same equivalence considerations which exist between the version derived from the methods of entropy-maximising, random utility, cost-efficiency, etc.. In addition we can modify the equations to make the attractiveness factors endogenous, as in the Harris and Wilson model (1978) and the static model can be embedded within a dynamic context.

All these extensions to the urban model of considerations relative to single submodel far from being accademic exercises are of considerable interest for two reasons:

a. the behaviour of individual submodels can be different when inserted in a global model and when they are considered separately because of the feedback which may occur. For example, the model of Lakshmanan and Hansen (1965), a production-constrained spatial interaction model, when introduced into a Lowry model behaves quite differently, as it becomes a model in which the production is in some complex way derived from the output of the model itself (Lombardo and Rabino, 1983);

b. the generalisations themselves can stimulate new thinking on the urban model. We can see in this book how Beaumont, beginning with a mathematical programming version of the Lowry model reformulated in incremental terms (variations in numbers of jobs, housing and services) created a model for the optimisation of urban developments.

2.4.2. From static linear urban models to dynamic non-linear models

This section focuses on current developments in urban modelling and on the contribution made by certain chapters of this book. The works referred to represent only a small part of the wealth of inventive thought distinguishing this field but allow us to identify the logical thread passing through the history of urban modelling.

The starting point of this analysis is of course the Lowry model (1964).

Although soon after its formulation it was considered above all a spatial version of the economic base theory (because of the emphasis placed on the causal aspect of the model) and for this reason also greatly criticised, its fundamental role in urban modelling history is linked to the basic "message" of the model: the city is a system, made up of a set of different states (associated with certain socio-economic quantities such as population and jobs, plus certain spatial elements) all interacting with each other through spatial and socio-economic interrelationships.

In this respect it can be said that the Lowry model has played in its own field a role similar to that of input-output models in the analysis of economic structures. This is not altogether surprising if we consider that the Lowry model is really a special kind of input-output model (Macgill, 1977b).

This similarity between the Lowry model and normal input-output models is true also for the way in which they deal with both economic and spatial inter-relations. Both treat them as being static and linear. That is true at least for Garin's matrix version of the model (1966) which is the one most frequently used. If we look closely at Lowry's original model we find in fact implicit elements of non-linearity, such as those associated with land-use constraints. Anyone who has worked with models of this kind will recall the complications these elements introduce into the resolution of an otherwise simple model.

The reference to the Lowry model recalls another famous model of the late 1960's, the dynamic urban model of Forrester (1969).

Without diminuishing its many positive qualities and the important role it played in introducing a dynamic view of urban systems we should underline that unlike the Lowry model which has a simple basic form but was capable of progressive and more refined extensions, the Forrester model (using Dynamo language), attemped somewhat presumptuously to deal in an elementary way with the whole complexity of the urban system (using for example numerical tables for complex functions) but did not stimulate refinements of the model itself. In fact, despite its initial success, the Forrester model has had relatively few applications compared with the great number of models originating from the Lowry prototype.

Many of the most important developments of the Lowry model were made by Wilson (1974). These can be divided into three groups. Those with:

a. a more rigorous foundation of the theoretical aspects of spatial interaction, with the introduction of stochastic elements;

b. a more general treatment of spatial interaction, with the introduction of non-linear elements;

c. a development of the Lowry scheme through the concept of disaggregations.

As far as the first group is concerned we find the use of the entropy-maximising methods according to which the observed interaction is the modal value of a multinomial probability distribution which is almost always discrete, limited by a given set of constraints corresponding to empirical evidence (Shannon and Weaver, 1949). This modal value is calculated using Lagrange multipliers:

$$\max_{\underline{\underline{x}}} \; Z = -\sum_i x_i \ln x_1 + \sum_n \sum_i \lambda_n \, g_n(x_i) \, , \tag{3}$$

where:

$\underline{\underline{x}}$ is the probability distribution;

λ_n is the Lagrange multiplier associated with the n^{th} constraint;

$g_n(x_i)$ is the n^{th} constraint function.

Apart from the acknowledged importance of a sound theoretical base and the fact that (3) is the basis for the construction and calibration of a great family of spatial interaction models (from the four elementary models to the variously disaggregate models from group c. above), equation (3) has also been the stimulus for other new theoretical interpretations of spatial interaction models. The main characteristic however which should be emphasijed here is that the underlying theory is developed from probability theory which is certainly more suitable for the analysis of economic and social phenomena than the deterministic (or analogic) approach which was used prior to this. Later we shall come back to the question of the superiority of the stochastic approach to the deterministic approach, observing for the moment simply that Wilson and many successive researchers undoubtedly failed to develop the full potential of the probability approach by taking into account only the mean (or mode) of the probability distribution and treating models as if they were deterministic (for example, in the treatment of structural stability problems).

As far as group b. above is concerned the non-linearity is clearly an improvement on the original model (linearity frequently entails approximations, especially as many urban phenomena such as congestion, saturation etc. are intrinsically non-linear). It appears in three types of phenomena: i) that associated with the relationship between flow size and attractiveness factor (eg. economies of scale), ii) that associated with flow constraints (eg. the family of models with different constraints at the origin and at the destination), iii) that where the attractiveness factor is in some way related to the flows arriving at the zone (a subset of the first type above). The most interesting is this last, described in the Harris and Wilson model (1978):

$$T_{ij} = A_i O_j D_j^{\alpha} f(c_{ij}) \quad \text{with} \quad D_j = k_j \sum_i T_{ij}, \qquad (4)$$

where:

T_{ij} are flows from i to j;

O_i is the constrained origin;

A_i is a normalizing factor;

D_j is the attractiveness term (a function of T_{ij});

$f(c_{ij})$ is the impedence of distance;

α e k_j are parameters.

It can be shown for certain values of the parameters that this model has multiple solutions, the number of which is also related to the changes in parameters. This multiplicity of solutions, which reflects the multimodality of the function (3), is one of the most noteable characteristics of non-linearity. It means that the function (3) is not strictly concave within its domain of definition, and it poses serious problems for the model resolution (Phiri, 1980). Its relevance is associated with the fact that it introduces differential topology (concerning the structural stability of systems and processes of the catastrophe type). This, as well as being of theoretical interest, is very important in planning (Wilson, 1981a).

As far as the models in group c. are concerned we should point out that the process of disaggregation, besides allowing a more detailed analysis of the urban system, is a process strictly associated with the entropy concept according to which each stage of disaggregation, corresponding to the introduction of new constraints in (3), is equivalent to an improvement in the description of the complexity of the system, from a Weaver II type system (disorganised complexity) to a Weaver III type system (organised complexity) (Weaver, 1958).

A successive stage of development, i.e. the passage from static non-linear models to dynamic non-linear models, is the model formulated by the Brussels school (Allen et *al.*, 1978). The innovative feature of this model was not simply the fact that it was dynamic (after all, many post-Lowry models were dynamic, see, for example, the TOMM model, Crecine, 1964), but that it was also non-linear.

In this model the activities, which are the same as those used in the Lowry model, evolve in time according to a non-linear logistic growth dynamic:

$$\dot{x}_i = \varepsilon_i [D_i - x_i] x_i , \tag{5}$$

where:

x_i is the activity x in zone i;
D_i is the carrying capacity for activity x in zone i;
ε_i is a proportionality factor.

The carrying capacity for an activity in a given zone is defined in function of the values of the other activities in the other zones according to economic and spatial relations of the Lowry type. There is a close relationship between this model and the Lotka-Volterra model (Volterra, 1927) of the growth of different interacting animal populations.

An important aspect of this model is that in the processes of spatial differentiation that is the occurrence of different possible urban spatial patterns (associated with catastrophic processes deriving from non-linearity) stochastic elements are considered. It is the random fluctuations of the system which, in proximity to the bifurcation points (in the evolution of the system itself), determine the path that will be followed.

Wilson was responsible for another important contribution to the development of dynamic non-linear models. This is the embedding in a dynamic context of the model of Harris and Wilson previously mentioned. Model (4) is considered the equilibrium state of a system which tends to move towards this state with a speed proportional o the distance from equilibrium:

$$\dot{D}_j = \varepsilon [\sum_i T_{ij} - k_j D_j] D_j^n , \tag{6}$$

where n may take different values such as -1, 0, 1, 2,. . . .

Model (6) is, like the preceding one, a logistic growth model of interacting populations and the function (3) from which it is derived plays in this case the role of "potential" function of the system whose gradient determines the dynamic: ($\dot{D}_j = \partial Z/\partial D_j$). It is important to note that an economic interpretation can be given to the dynamic process which arises from the disequilibrium between demand $\sum_1 T_{ij}$ and supply $k_j D_j$.

Wilson in this book investigates the potential of this kind of dynamic approach very full, suggesting ways of achieving a more refined analysis of the demand and, in particular the supply side and a more extensive application to a number of subsystems as well as to the urban system as a whole. One of these suggestions, of particular interest here, is the application of a model like (6) to the transport system, as discussed in Wilson (1983).

A totally different approach is offered by the model of Leonardi and Campisi (1981), even though like Wilson's model it uses microeconomic aspects of spatial interaction, or more precisely random utility theory.

Leonardi and Campisi obtain the following expression for the transition rates (which are non-linear and non-constant) between different zones of a spatial system:

$$r_{ij}(t) = \lambda [Q_j - P_j(t)] f_{ij} e^{-\beta [V_j(t) - V_i(t)]} , \tag{7}$$

where:

r_{ij} is the transition rate from i to j;

Q_j is the carrying capacity of zone j;

P_j is the population of zone j;

V_i is a measure of the utility of staying in zone i;

f_{ij} is an impedence function of distance;

λ and β are parameters.

Associated with (7), the following conservation equation gives the levels of $P_j(t)$ in the various zones:

$$\dot{P}_j(t) = \sum_i P_i(t) r_{ij} - P_j(t) \sum_i r_{ji} . \tag{8}$$

The non-linearity of (7) derives from the term V_i associated with utilities, which are complex functions of the populations P_i and, given (8) also of the rates r_{ij}, through the differential equations of the type:

$$\alpha V_i - \dot{V}_i = a_i + \frac{\lambda}{\beta} (\Phi_i - \Psi_i) , \tag{9}$$

where:

Φ_i is the total accessibility in i at $(Q_j - P_j)$;

Ψ_i is the potential of the population in i;

α and a_i are constants.

Leonardi in this book explores all the aspect of the model described above from its derivation (from random utility theory) and "catastrophic" characteristics (stability, bifurcation, etc.) to its application to residential mobility (and to joint-residential mobility) also discussing its possible further developments.

One development of particular interest is the relaxation of the assumption that the carrying capacities Q_j are constant and the modelling of their evolution. Some indications on how to proceed are contained in the IIASA Research Programme on "Nested Urban Dynamics" (Johansson, Korcelli, Leonardi, Snickars, 1983), an important research project with the aim of developing urban modelling, both theoretically and experimentally, exploring many of the aspects which in this section are considered important for future progress.

The above problem of carrying capacities is interesting and stimulating as it involves the study of interaction between dynamic processes with different speeds of change (eg. the dynamics of stocks, i.e. of carrying capacities, and the dynamics of activities). In this respect the above models of Wilson and Leonardi represent two extreme cases.

Wilson's model considers local dynamics (eg. stock dynamics) interacting through static spatial interaction models which are the equilibrium solution of a dynamic process (eg. activity dynamics) which is so fast that the equilibrium assumption is reasonable.

Leonardi's model takes the dynamics of flows (eg. activity dynamics) occuring in constant "containers", which represent the state of a dynamic at a given moment (eg. the stock dynamic) which is so slow that it is reasonable to assume it is constant.

In this respect the study of dynamic processes of different speeds can also be seen as the problem of the integration of these two models.

Leonardi's model in particular can be considered a recent evolution of the multi-state models of population (Rogers, 1975), a particular kind of compartmental model. These models originating from the model of Leslie (1945) (with constant transition rates) have gradually become more complicated, first with systematic disaggregation (multi-regional and multi-state models) then with the introduction of non-linearity (Ledent, 1978, Okabe, 1979, Sheppard, 1983c). Even though Leonardi's model is basically conceived in probabilistic terms this aspect is not fully developed in its treatment. This is also true for the other models discussed above. A more complete stochastic treatment is found in the model of Weidlich and Haag (1983), where the differential equations are expressed in terms of probability distributions of different states (even though only the mean value of the distributions are then treated analytically). The model is also a multi-state model in which the transitions are determined from the non-linear interaction of the popoulations in different states.

An even more complete stochastic treatment is found in the model of Sidkar and Karmeshu (1982). This is a compartmental, multi-regional demographic, urban model with non-constant linear rates of transition, derived from the Okabe model (1979) referred to previously. An equally complex treatment is in the Monaco and Rabino model (1984) of interacting populations with constant non-linear rates of transition. Both of the models consider not only the mean but also other moments in the stochastic process.

All the above points to stochastic analysis as being one of the most promising aspects of research on urban models. Certain progress has already been made but much remains to be done. De Palma provides in this book a very theoretical contribution on the state-of-the-art of compartmental systems (both deterministic and stochastic approaches) also describing the theorems of Lehoczky (1980) and Kurtz (1978) which define the conditions for which the results obtained from the deterministic approach continue to hold even in a stochastic context.

From this contribution the superiority of the stochastic approach emerges clearly and constitutes a stimulus to proceed in this directions.

2.5. *Urban form and transport*

2.5.1. Introduction

A common element running through the preceding analyses, though not explicitly highlighted, concerns the effects which interrelationships between the location of socio-economic activities and transport have on the structure of urban space. More exactly it concerns the way in which location behaviour of socio-economic activities determines the urban form in function of a given transport network and how, in its turn, the transport network is structured in function of a certain pattern of socio-economic activities (i.e. in function of the urban form). In this section the analysis is therefore carried out at a more general level, looking in particular at the relationships which exist at an aggregated level between urban patterns and transport (*).

2.5.2. Spatial structure and transport: state-of-the-art

There have been two completely different approaches to this subject. One is an economic approach, the well-known "New Urban Economics" (Mills and Mackinnon, 1973), and the other, which could be defined a functionalist approach, consists of the spatial interaction models (Wilson, 1974).

(*) The terms "spatial structure" and "urban form" are used indifferently in the present discussion.

New Urban Economics approach

We define as New Urban Economics (NUE) any systematic theoretical explanation of the urban spatial structure based on neo-classic economic principles, which describes the structure of urban space as a result of a market process.

The precursors to NUE were two theoretical paradigms – von Thünen's theory of rent (von Thünen, 1826) and Lösch's central place theory (Lösch, 1940) (*).

In the development of NUE we can distinguish two modelling phases (cf.: Anas and Dendrinos, 1976).

I. The first phase

The first phase, which occured mainly in the sixties, saw the extension of von Thünen's model to the urban context. Fundamental contributions were made by Beckmann (1957a, 1969), Alonso (1960, 1964a), Muth (1961, 1969) and Wingo (**).

A common characteristic of these works is the use of a utility-maximising approach through which the individual's choice of location and decision on the amount of urban land to be consumed are examined.

The underlying hypotheses of this approach can be summarized as follows:

a. the city has a single centre, the central business district (CBD), in which all the productive activities are concentrated, and an external ring in which all the residents are located. Space is considered uniform i.e. homogeneous and isotropic;
b. perfect competition exists between individuals, whose behaviour is rational and based on perfect knowledge of the market. In addition, at equilibrium, demand is always satisfied and supply completely consumed.

The approach assumes that individuals maximise the utility associated with the goods and services to be consumed, subject to the condition that their level of consumption is limited by a given available income.

(*) Other than the authors cited here other precursors of NUE were those from the Chicago school (Park, Burgess and McKenzie, 1925, Hoyt, 1939, Harris and Ullman, 1945) and certain other urban economists (Hurd, 1903, Haig, 1926, Ratcliff, 1949).

(**) There has been a vast amount written on NUE but for reasons of space we have limited our references to those which are felt to be fundamental. For a more complete review cf.: Richardson (1971, 1977b), Mills, Mackinnon (1973), Anas and Dendrinos (1976).

In general utility is defined on the basis of:

a. the quantity of consumption goods produced in the city (in the CBD) and available in location d;
b. the residential services offered in location d by housing suppliers;
c. unit prices, of consumption goods (which do not vary spatially) and residential services in location respectively d;
d. transport costs associated with location d (which constitute a negative component of utility) (*).

For the individual, maximising utility means, first of all, determining his optimum residential location (in terms of distance from the CBD) given transport costs and housing prices (which constitute the spatial problem) and then, in his optimum location, selecting the optimum set of goods and services which can be consumed given his total available income.

If h(d) are the residential services (housing) in a location d and p(d) are the relative prices, the condition that expressed the equilibrium location of the consumer is:

$$\frac{h(d)\partial p(d)}{\partial d} = -p_e(d)t, \tag{10}$$

where: $p_e(d)$ is the monetised value of the marginal utility of leisure time (or, in other words, the opportunity cost of travel time) and t is the travel time.

(10) describes the trade-off relationship which must exist between housing prices (rents) and distance (transport costs) so that the consumer is in equilibrium and his location is optimum.

From (10) we find that $\partial p(d)/\partial d < 0$, i.e. that a reduction in residential prices (rents) is necessary to compensate the increase in travel costs as distance from the CBD increases.

For a stable equilibrium in the city, all individuals, if characterised by the same

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(*) In general the first phase NUE models it is assumed that transport costs increase more slowly than (or at the same speed as) distance (the elasticity of transport costs in relation to distance is less than or equal to 1). This hypothesis is necessary in order to arrive at the results on price structure and density. The relaxation of this hypothesis and its consequent implications on urban form are discussed by Papageorgiou (in this book).

income and same preferences, must have the same level of utility in the different locations. Thus the rent gradient $(\partial p(d)/\partial d)$ will have a monotonically decreasing curve, as shown in fig. 1 (*).

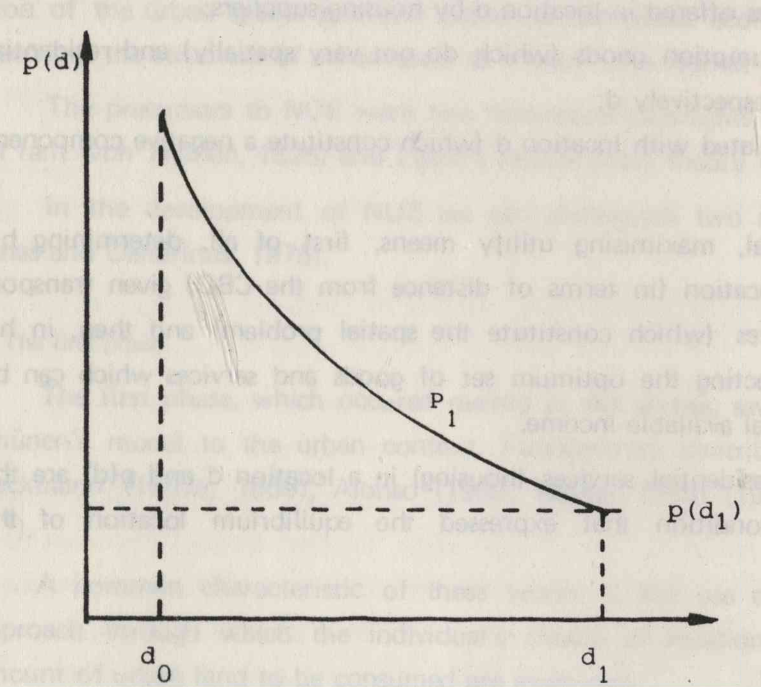

Figure 1 – Rent gradient for consumers with identical income

In addition, since p(d) decreases as distance increases and utility remains constant as prices and locations vary, housing consumption, h(d), increases with distance $(\partial h(d)/\partial p(d) > 0)$. In order to have spatial equilibrium in the whole city the above results must be valid for all radii from the CBD. That is, at a given distance from the CBD, residential prices must be equal.

From the above we obtain a result which is fundamental to the NUE approach, that given the assumptions of uniformity, homogeneity and isotropy of space, the urban form will be circular.

(*) To determine the rent gradient see fig. 1 in which the curve has been traced between d_0 (radius of CBD) and d_1 (radius of the city). As the cost of housing in d_1, $p(d_1)$ is known, being equal to the cost of production of housing on agricultural land, and all individuals have the same utility, the value $U(d_1)$ can be determined and defines the level of utility in the whole city. For any location of d_i it is therefore possible to determine value of $P(d_i)$ as U and the associated cost of transport are known.

This holds even in the case where individuals are not identical for reasons of taste, preference or income. In this case the bid-rent (bid-rent functions) of different consumers are considered (see fig. 2). The curves (F_1, F_2, F_3) represent the hypothetical price, which for a certain level of utility (U_1, U_2, U_3) a consumer would be willing to pay in different locations so as to be indifferent about these locations.

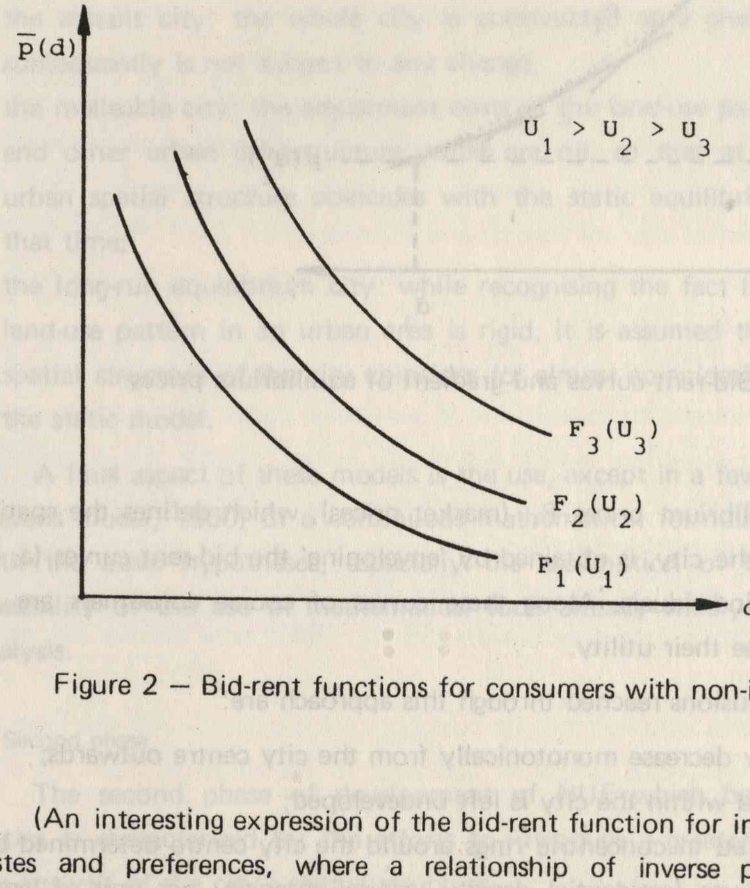

Figure 2 — Bid-rent functions for consumers with non-identical incomes

(An interesting expression of the bid-rent function for individuals with identical tastes and preferences, where a relationship of inverse proportionality emerges between utility and bid-price is described by Beckmann in this book).

In general the bid-rent curve or, more specifically, the gradient of this curve changes as the income of consumers varies (i.e. h(u) and $p_e(u)$ change with income). If, for example, we assume that the quantity of residential services consumed, h(d) increases with respect to the marginal value of leisure time $p_e(d)$ as income increases, we find that the inclination of the bid-price curve increases (i.e. the gradient becomes less steep) (cf.: fig. 3).

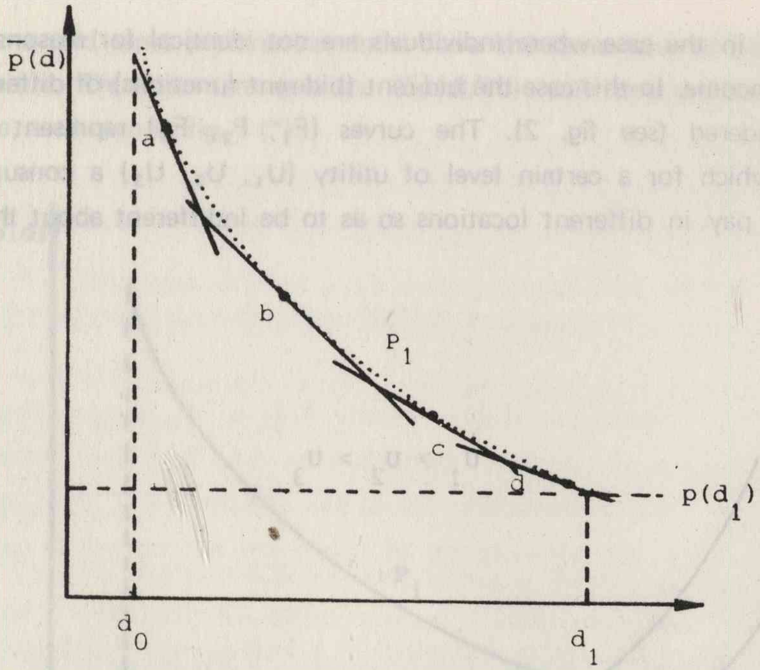

Figure 3 — Bid-rent curves and gradient of equilibrium prices

The curve of equilibrium prices P_1 (market prices), which defines the spatial equilibrium pattern of the city, is obtained by 'enveloping' the bid-rent curves (a, b, c, d) of the different individuals. Along these curves of course consumers are in locations which maximise their utility.

Some general conclusions reached through this approach are:

a. land rents and density decrease monotonically from the city centre outwards;
b. no portion of the land within the city is left undeveloped;
c. land uses are structured in concentric rings around the city centre determined by the intensity of land-use (residential density, capital intensity per unit of land) which decreases monotonically outwards from the city centre. Hence, two different land-uses cannot be found at the same distance from the centre.

These conclusions at which all of the first phase NUE studies generally arrive, constitute axiomatic characteristics which define the overall spatial configuration of the city and hence its form.

From the methodological point of views, the essential characteristic of NUE models, and in particular those of the first phase, is the static nature of the approach. It is assumed that the equilibrium configuration of the city (derived from the bid-rent mechanism) does not depend on past configurations nor on expectations of possible future forms and that, in addition, market land prices do not influence in determining that form.

To further clarify the static nature of this approach it may be useful to mention three different definitions of the city derived from the above assumptions (Fujita, 1983):

a. the instant city: the whole city is constructed at a given moment in time and subsequently is not subject to any change;
b. the malleable city: the adjustment costs of the land-use pattern (i.e. building costs and other urban infrastructure costs) are nil, so that at each time interval the urban spatial structure coincides with the static equilibrium spatial structure at that time;
c. the long-run equilibrium city: while recognising the fact that in the short-run the land-use pattern in an urban area is rigid, it is assumed that in the long-term the spatial structure of the city coincides (or almost coincides) with that predicted by the static model.

A final aspect of these models is the use, except in a few cases (eg. Herbert and Stevens model, 1960) of a continuous mathematical formulation. This is connected with the basic hypotheses, especially the assumption of uniform space and the possibility of the use of mathematical tools already widely employed in economic analysis.

II. Second phase

The second phase of development of NUE which began at the end of the sixties is characterised by the efforts to introduce a greater degree of realism into the analysis of the city's spatial structure.

It was recognised that the axioms used gave an excessively simplified and largely unrealistic view of the city. In fact the spatial structure cannot be reduced in such a simple (or simplistic) way to a monocentric and circular form but tends in fact to spread in an "oil spot" fashion and have an irregular outline. The land rents and densities often increase in proximity to certain centres in the urban space as there are poles of concentration of population and activities. Lastly, at a given distance from the city centre it is likely that a mixture of land-uses will be found.

Thus it is obvious that the structuring of urban space results from factors and mechanisms far more complex than those so far considered in the analysis. It emerges above all that urban structure is the result of the interacting behaviour of a number of actors (households, manufacturers, public bodies etc.). These interactions take place in a non-uniform space, the differentations of which are to a certain extent both determined by and the cause of the interactions themselves.

The principal new elements of the second phase are:

a. the greater attention paid to the existence of externalities in urban space. The formation and effects of externalities are recognised as playing a determinant role in the spatial structure of the city (cf.: Papageorgiou, 1983, for a review). For example, it is made clear that the presence of spatial differentiations in an urban area such as poles and environmental quality can create effects of agglomeration or increase in utility and thereby generate increasing rent functions. (For an analysis of the polycentric city see Casetti and Papageorgiou, 1971, Papageorgiou, 1974, 1976b, and for an analysis of environmental effects see Papageorgiou, 1976b and his chapter in this book).

   The existence of economies of scale and external economies can induce individuals (firms and individuals) to concentrate in particular zones of the urban space until the effects of congestion, which eventually occur, nullify the advantages produced by concentration, (cf.: Lave, 1970, Mirless, 1972, Dixit, 1973).

   It is the analysis of congestion which is of particular interest in this phase of development. From the acknowledged fact that congestion makes the city more compact, limiting its spatial extension (Strotz, 1965) later studies (eg. Mills and de Ferranti, 1971, Solow and Wickrey, 1971, Solow, 1972) come to the conclusion that land area destined for the transport infrastructure and the level of congestion decreasing functions (linear or concave) of the distance from the city centre;

b. the efforts made to introduce the time dimension more explicitly. Even though these efforts have been channelled principally into comparative statics approach, the main aim of which being to analyse the effects that changes in income, population and transport costs produce on the urban spatial equilibrium (and, in particular, bid-rend curves), it was stimulated by the recognition that the city is by nature a dynamic reality.

   We can distinguish two kinds of approach to the analysis of the dynamics of the spatial structure of the city (cf.: Miyao, 1981) which are however closely interconnected. They are: (i) the analysis of the stability of the equilibrium of a static spatial configuration (stability analysis) and (ii) the analysis of the evolution of the spatial configuration (growth analysis). Interesting examples of these two types of approach are given in this book by Papageorgiou, even though as far as the second type is concerned he seems to be relatively entrenched in the comparative statics approach. Arguing that at an aggregate level the urban spatial structure can be considered to be in a state of dynamic equilibrium, Papageorgiou examines how the urban form is determined and evolves in time. More precisely, he examines the variation of dimension (population and utility) and the form of the city (land values, quantity of land consumed and density) relative to changes in income, technological level, agricultural land prices and interaction costs in the urban area. It is therefore, like Beckmann's contribution, a general analysis of equilibrium, in which the aim is to find out to what extent variations in the socio-economic structure of the city influence the form of the urban space and vice versa.

In this respect Papageorgiou's work belongs indisputably to the second phase of development of NUE to which it provides a very valuable contribution.

Even in this second phase the dynamic analysis of the urban spatial structure appears relatively little developed, due above all to the difficulty of introducing both time and space into a continuous model as it involves an operation of simultaneous integration (cf.: Pines, 1976).

One way of resolving this problem is to introduce discontinuities into the continuous model like those which characterize the discrete model. Three types of model are therefore possible (cf.: Richardson, 1977b):

1. models discrete in terms of space and time (for example, Herbert and Stevens, 1960, Ripper and Varaiya, 1974);
2. models which are continuous in space but discrete in time (for example, Pines, 1976, Anas, 1976);
3. models which are continuous in time but discrete in space (for example, Hochman and Pines, 1973).

A last approach is to develop non-spatial models of urban growth (eg. for the residential sector), which can provide nevertheless useful elements for the analysis of evolution of land-use patterns over time (for example: Evans, 1975, Muth 1976).

Although a truly dynamic model of the urban spatial structure is yet to be developed, the studies referred to above represent some first and promising steps in that direction. Further advances, although not yet sufficiently consolidated nor completely operational (in that they are not ready for immediate empirical testing) can be found in Papageorgiou (1980, and in this book), Miyao (1981) and Fujita (1983).

The functionalist approach

Unlike the economic approach, the functionalist approach — as the spatial interaction approach has been called — draws its concepts from the observation of empirical regularities which manifest themselves in the form of spatial interdependencies in the distribution of socio-economic activities in urban space.

These interdependencies appear in the intensity of flows of goods and people between the various activity locations — the flows being more intense when the generation (attraction) capacity of activities is greater (due for example to a larger land area occupied) and the distance between locations is less.

Unlike the NUE approach where there is a direct relationship between the mechanism of distribution of goods and people in the urban space and the resulting structure of that space, in the spatial interaction approach this relationship is less immediate and not so axiomatic. The urban form is seen as the result of a set of interactions of goods and people occuring in an urban space which is not conditioned or predetermined by any hypothesis, but which is essentially a function of the interdependence observable in the spatial distribution of activities (hence the use of the term functionalist). Thus, given the structure of interactions in an urban area, one can draw a 'map' of the accessibilities (or potentials) of that area which can be interpreted as a representation of its spatial structure.

From Reilly (1931) to Hansen (1959), Lowry (1964) and Wilson (1970a, 1971, 1974), to mention only the most outstanding contributions, the spatial interaction approach has been widely applied in urban and regional studies, not so much for the analysis of spatial structure but for the location of activities.

The theoretical and methodological refinements, in particular the introduction of entropy-maximisation (Wilson, 1970a) and the use of various forms of utility-maximising models (Williams, 1977) in the seventies, gave new potential for the explanation of urban form, making it possible for not only spatial but also economic interactions to be described (*). We are thinking here for example of the

(*) The relationship between the entropy version and the economic version (in particular the random utility version) will be illustrated in 3.1..

improvements made in the formulation of terms for expressing potential and accessibility (Williams and Senior, 1978, Leonardi, 1979, Wilson, in this book).

From the point of view of the analytical formulation, two well-known features characterize the spatial interaction model.

a. Flexibility: the relative case with which one can go from simple to more complex formulations, by disaggregating for example the variables of the model by household type, housing type, means of transport etc. (Wilson, 1974), by incorporating more complex attraction terms (see the retail services model of Harris and Wilson, 1978) or principles of individual economic behaviour (Coelho and Williams, 1978, Coelho, 1979), by integrating several spatial interaction models of subsystems in a general urban model (Lowry, 1964, Wilson, 1974 and in this book) or by integrating different transport models in a system of integrated transport models (Wilson, 1974, Wilson et *al.*, 1981), all make it possible to analyse specific problems such as the "efficiency" of the spatial structure from the energy point of view. Beaumont in this book for example, points out that the calculation of the energy efficiency of a spatial structure should be an integral part of the definition of land-uses and the overall journey patterns and not an ex-post operation made after the model has been run or simply based on transport costs minimisation, as is generally done in energy studies. In this respect the problems which must be taken into account are: (i) the treatment of the problem at the micro-level, implying the consideration of the individual's decision process (see, the mathematical programming version of Lowry's model, by Coelho and Williams, 1978); (ii) the explicit consideration of the existing infrastructure (cf.: Beaumont and Keys, 1982 and again Coelho and WIlliam's model, 1978); (iii) the inclusion of variables in the analysis of energy efficiency other than those relating to journeys (i.e. consideration of variables which take account of the in-place energy consumption of the different activities cf.: Beaumont and Keys, 1981); (iiii) the use of a dynamic approach.

b. Ease of operation: the fact that the spatial interaction model is relatively simple to apply (probably easier than the economic type model) makes it a useful aid to planning. For a discussion of this point see Batty, 1979, Webber, 1981, Wilson, in this book.

The introduction of the time dimension into the functionalist approach (Wilson, 1976a) occured more or less at the same time that a dynamic version of the NUE model was being experimented with, i.e. around the second half of the seventies. Mathematical techniques belonging to system dynamics, especially catastrophe theory (Thom, 1972) and bifurcation theory (Jordan and Smith, 1977), showed themselves to be fundamental tools for this kind of analysis. For a review of these see Beaumont, Clarke and Wilson (1981a), Wilson (1981c) and Beaumont (1982). In this book Beaumont suggests that Q-Analysis (Atkin, 1974, 1981) which is a special technique for the dynamic analysis of structural inter-relationships, can be a useful tool for describing the evolution of the relationship between behaviour and structure of a system. He shows how, by developing the existing connections between bifurcation theory based on differential topology and Q-Analysis based on algebraic topology it is possible to achieve a better description of structural changes in a dynamic system.

Compared with NUE models, dynamic spatial interaction models (even if they are partial) are relatively easy to operate and have probably been applied more widely especially in simulation experiments (for example, Wilson and Clarke, 1979, Lombardo and Rabino, 1983a).

As already mentioned Wilson's chapter in this book is particularly relevant to dynamic analysis, showing how by introducing supply and competition between demand and supply in a simple spatial interaction model, the system can produce different and unexpected behaviour (eg. multiple equilibrium solutions, oscillations, instability etc.) even for very small variations in the parameters. His chapter therefore contains all the basic elements for the analysis of dynamic processes in the structuring of urban space, although many of these elements incorporate concepts and mechanisms belonging to the economic approach.

2.5.3. Conclusions

New Urban Economics and spatial interaction models constitute two alternative but complementary approaches for the analysis of the relationships of transport and urban form especially at an aggregate level.

We have tried above to highlight the essential features theoretical and methodological of the two approaches making reference to the contributions in this book which are relevant to the subject. We now discuss in which directions research could proceed in the future and where we feel there lies undeveloped potential.

First of all we deal with theory and methodology and then briefly mention some operational aspects.

Any advance in theory and methodology of the analysis and interpretation of a phenomena will be rooted in the existing and already consolidated body of studies.

We therefore look first at possible developments of the two main approaches above, and then at developments which may ensue from their integration with approaches from outside the immediate field.

The following aspects of the NUE and functionalist approaches seem to offer scope for useful future research.

I) Analysis of externalities. As we have seen previously, the phenomenon of externalities in the urban area is one of the questions most widely discussed at present, especially in New Urban Economics. Although a considerable number of theoretical and applied studies exist on the effects of externalities (cf.: Papageorgiou, 1983) there is still a need for an analysis of the processes of their formation, In particular of externalities deriving from agglomeration processes.

We argue that the NUE approach provides the fundamental conceptual apparatus.

As the formation of externalities is closely connected with the growth of the urban structure, an explanatory theory of the formation of externalities requires the explicit consideration of the time dimension and therefore involves the analysis of the dynamics of spatial structure.

The work of Miyao and Shapiro (1979), Kanemoto (1980a, 1980b), Miyao, Shapiro and Knapp (1980) and Miyao (1981) are particularly promising in this respect.

II) Partial versus global analysis. Full comprehension of the relationship between urban form and transport requires a global approach which takes account of all the components and interactions in the system. However, most existing studies, both economic and functionalist, are partial analyses which concentrate in general on the analysis of the interactions between two subsystems at most (usually transport and one other). There are many comprehensive studies which although still cumbersome and not as systematic as models are nevertheless promising for future development. In this respect the descriptive function of urban morphology obtained by Papageorgiou (see equation 30 in his chapter in this book) and the formulation of comprehensive or integrated models of the urban system by Wilson (in this book) and Nakamura, Hayashi and Miyamoto (1983) deserve mention.

In most existing formulations, the relationship between transport and urban form has been explored unilaterally. In general they concentrate on the analysis of the implications of transport facilities (measured in terms of distance, travel costs, travel times etc. which are given exogenously) on spatial organisation, or else the analysis of the effects on the transport structure of a given spatial form (measured in terms of jobs, population etc. also given exogenously). Thus an important improvement could be obtained by refining the modelling of these interrelationships. This would mean a close integration of transport submodels with other submodels (in both partial and global analysis), by making the determination of travel costs endogenous and introducing the supply side into the transport system, as suggested by Wilson (1983), or (for a given transport cost function) as illustrated by Puu (1979b).

III) Dynamic analysis. The need for a dynamic approach in the analysis of urban form has already emerged many times in the course of this discussion and clearly constitutes one of the priorities for future research.

A dynamic analysis is fundamental in order: (i) to give a full interpretation of socio-economic processes which over time have produced a given spatial structure; (ii) to provide elements for the evaluation of the efficiency of those processes; (iii) to suggest ways and means of controlling them; (iiii) to delineate possible future configurations of the spatial structure and the various directions of development which are likely to produce those configurations.

Although dynamic models are a relatively recent development in this field, a certain number of studies have already reached a relatively advanced level (cf.: Papageorgiou, 1980 and in this book, Wilson 1981a and in this book).

Of the NUE type studies, the work of Anas (1978a), Mills (1981), Miyao (1981), Wheaton (1982) and in particular Fujita (1976b), Dendrinos (1981a) and Fujita (1983) are excellent starting points. For example Fujita (1976b) presents a dynamic version of Alonso's model and Fujita (1983) shows how the dynamic modelling of future land-use price expectations, can justify the existence of high land prices in the outskirts of the city and help to explain the 'oil spot' effect. Dendrinos (1981a) uses structural stability analysis (Thom, 1972, Zeeman, 1977) to examine from the qualitative point of view the dynamic processes which determine the evolution of the urban form.

Among the functionalist studies the various works of Wilson (1978c, 1981a, 1983 and his contribution to this book) highlight a number of topics on which future research efforts could well be focused. The most important are the following: (i) the introduction of supply, and suggestions on how it can be modelled dynamically in particular in single-constrained spatial interaction models; (ii) a methodological framework for the building of dynamic comprehensive models of the urban system; (iii) suggestions for the use of the dynamic approach for exploring the formation of new spatial configurations leading towards a study of morphogenesis of the spatial structure (see point IV later).

With respect to the implications of the energy crisis on the organisation of the human environment (Beaumont and Keys, 1982), dynamic analysis offers great potential for the determination of an energy-efficient spatial structure and the evaluation of the impact of alternative energy policies (Beaumont, in this book).

As already mentioned above, important stimuli for the analysis of the relationship between urban form and transport have come also from other disciplines.

In the field of urban and regional science the use of models taken directly or indirectly by analogy from other branches of the physical and human sciences has been fairly common and has often contributed considerably to progress in theory and methodology. Recently systems approaches from biology, ecology, chemistry and economics in particular have been applied successfully to socio-economic and spatial systems.

We therefore add to the three points above a further two containing the elements from these other disciplines which appear to offer the most positive stimulus to future progress.

IV) From biology, ecology and chemistry the most important contribution seems to be the dynamic aspect of systems analysis (see Wilson, 1981a, Beaumont, 1982, for a review).

There have been interesting biological studies on the evolution of highly complex systems and on the evolutionary behaviour of organisms and systems in interaction with their environment (cf.: Maynard Smith, 1978, Mayr, 1978). In the field of ecology interesting work has been done on competition between species (cf.: May, 1978, and for a review Jørgensen, 1983). Some applications of ecological concepts to urban analysis can be found in Dendrinos and Mullally (1981a) in which Lotka-Volterra's prey-predator model is applied to population growth in some North American cities, and in Wilson (1981a) where a spatially disaggregated version of the above model is presented. Although the full implications of the application of this kind of approach to spatial structure have still to be assessed it is clear that the analysis of human behaviour in space can be of help in understanding the spatial structure which derives from it.

The work of Prigogine and his school in the field of chemistry (cf.: Nicolis and Prigogine, 1977) on the evolution of dissipative structures offer new ideas on how interdependence of variables may result in the self-organisation of the system itself, where new structures and organisation can be generated or destroyed as the system evolves. The application of these concepts to the analysis of spatial evolution and in particular urban growth can help to explain and predict, above all from a topological point of view, the appearance of new nuclei (and disappearance of old nuclei) for example and can therefore be useful in exploring the formation of new spatial configurations (Allen et *al.*, 1978, Allen and Sanglier, 1979a, 1981b) (*).

V) New economic approaches. Recently some new theoretical and methodological developments have been made by neo-Marxian economists offering interesting alternatives to the more traditional neo-classic approaches to spatial problems.

Whereas in the past Marxian analysis mostly consisted of an interpretation of spatial problems couched in Marxist terms and strongly influenced by ideology, an effort has been made in the last ten years to produce a more general interpretation which is more consistent with spatial analysis (cf.: Lefebvre, 1972, Castells, 1973, Harvey, 1973, Shoukry and Scott, 1981).

This effort has concentrated recently on the development of a theory of production and accumulation (containing elements of Marxian, Ricardian and Keynesian thought), which comes in answer to the criticism that the neo-classic production function was inconsistent when a disaggregate formulation was considered (Sheppard, in this book).

The approach which is based on the model of Sraffa (1960) has already been applied in geography in the work of Scott (1976) who shows the interrelationship which exists between Sraffa's model of the production process and von Thünen's rent model and the possibility of their integration. Scott (1978) also analyses the impact of investment in transport on profit, salary and rent distribution in an urban area and later (Scott, 1980) presents an application of this approach to urbanisation and planning processes.

(*) With reference to compartmental models de Palma and Lefévre in this book present a survey of the most recent developments and the possibility of application to urban and regional science of dynamic approaches formulated in other scientific disciplines.

The article by Sheppard in this book in which he proposes a spatial extension of Sraffa's model (the Morishima version, 1973) using the theory of geographical potentials is particularly promising in this connection. Although his approach has been developed at a macro (regional) scale, it could be extended to the urban scale, introducing the consideration of stock (disaggregating capital) and the mechanism of formation of urban rents. This approach offers new potential for explaining the structure of economic interactions in the city and the resulting spatial implications in more realistic terms. In addition the social dimension underlying this approach constitutes a new point of view from which to tackle the problem of externalities.

VI) Operational aspects. In contrast to the considerable amount of effort made to develop the theoretical aspects of analysis of the relationships between transport and spatial structure the problem of the practical application of the models produced has been relatively neglected. It is obviously desirable that the functional relations that describe these relationships should be not only recognised but also quantified.

It emerges that the New Urban Economics models, although providing a sound conceptual base for defining the relationships are particularly difficult to operate when it comes to experimental verification. The spatial interaction models on the other hand have proved to be rather more easily converted into operative tools and more easily used in planning.

This results from the nature of the two approaches – more markedly orientated towards theory and methodology in the NUE studies, more orientated towards experimentation in the spatial interaction models – and the characteristics of the analytical formulation – in continuum in NUE models and in discrete form in spatial interaction models. Greater difficulty in quantifying variables has been encountered by the NUE models, and has led to the use of simplifying but unrealistic assumptions.

However there seems to have been a recent tendency for NUE models to adopt forms which can be more easily tested empirically (see, for example Papageorgiou's chapter in this book and Fujita, 1982b) and for functionalist models to incorporate economic principles of behaviour (cf.: Coelho and Wilson, 1976, Wilson, 1981a, Lesse, 1982, Wilson, in this book).

How relevant the complete integration of the two approaches would be is difficult to say. There remain many theoretical problems still to solve (for example that of interaction between micro and macro-level, or that of the aggregation of individual behaviour). What is clear however is that in both approaches, there are important advances to make both in the theory and methodology as suggested above in points I) to IV) and in the practical application of models. It is to be hoped that feed-back from application to theory will help to refine the conceptual aspects and may eventually contribute to such an integration, or at least the development of a more rigorous theoretical base and more operational models.

3. Main problems in theory and methodology: recent developments

3.1. *Relationships between the different approaches to the modelling of demand behaviour*

3.1.1. Introduction

The most important demand models in a location transport system are those involving consumer choice between alternatives, which are usually spatially differentiated. We are particularly interested in choice models for trip destination, for a route in a network and for the transport mode.

Apart from the specific content, which varies from case to case there is a fundamental similarity in the structure of these models in that all envisage a situation of choice between a discrete set of alternatives (of destination, route or mode) and measures of distance or cost which reduce the possibility of using them.

A first distinction can be made between static models of choice and dynamic models of choice.

Even though in the following we concentrate exclusively on comparative analysis of static models, we should specify that the exclusion of dynamic models is only for the following reasons. First, the literature on dynamic choice models is very limited and sporadic and therefore insufficient to carry out a real comparative study. Secondly, it can be argued that dynamic choice behaviour is simply a sequence in time of locally static choices, and therefore the basic mechanism is in fact the same as that of the static models we examine here. What differentiates dynamic choice models is essentially the fact that the attributes of the alternatives (in general expressed in terms of a measure of utility) are not constant, but vary in time as a function of the interactions between individuals and also the interactions between individuals and the physical environment in which the choice is made. Typical examples of attributes which vary because of these interactions are limited capacity (for example of the housing stock or a road link), prices, and all the negative externalities deriving from competition between different individuals for the use of limited alternatives.

However, while we acknowledge that the introduction of dynamics is one of the most interesting challenges for future research, we observe that this dynamic quality has more to do with the analysis the different attributes of alternatives than with the basic mechanism that produces the evaluation and choice. This mechanism, which is locally static, is the subject of the following analysis and is the fundamental element underlying all choice models, whether static or dynamic.

A further classification can be based on the distinction between aggregated models, based on observed phenomena at the macroscopic level, and disaggregated models, based on explicit assumptions (in general micro-economic) relative to the process of individual choice.

Among the former are included the models based on entropy maximising and the cost efficiency principle. Among the latter we have the classic models of utility maximisation typical of urban economics, and those based on random utility theory.

What is surprising is that such different and apparently conflicting theoretical assumptions lead to almost identical models. More precisely, we find that all the approaches which introduce effects of random dispersion on choice – i.e. entropy-maximising, cost efficiency and random utility – produce, under relatively weak assumptions, the so-called multinomial logit model.

In the following we analyse the theoretical aspect of these similarities and the relationships which make it possible to map one approach into the others.

3.1.2. Entropy-maximisation and cost efficiency

The equivalence between the two principles has been proved in the contribution of Smith to this book, to which the reader should refer for technical details.

Here we mention only the main features of the two methods and their equivalence.

It should be added that Smith applies the principle to an assignment problem in a congested network, but obviously it can be extended to any problem of choice between discrete alternatives. For this reason the comparative analysis of the two principles can be based on the simplest problem of choice between discrete alternatives, which is the following.

Let us take a population of P consumers and a set of alternatives j, $j = 1, \ldots, n$. Associated with each alternative j is a real number v_j which we can call the utility of alternative j for the population considered and which measures the attractiveness or relative advantages associated with the choice of j. In many applications v_j will consist either of a term for the cost of access to alternative j (eg. cost of transport) or a term for the specific attractiveness of j (eg. the dimensions of the shopping centre capacity etc.). However, from the theoretical point of view, which is what interests us here, such a distinction and disaggregation is not relevant, so we can consider a single numerical value to v_j which will be referred to as the utility of alternative j.

The way in which the entropy-maximising method as proposed by Wilson (1970a, 1974) deals with the problem of determining the distribution of choice between the various alternatives is well-known. Here we give a brief reminder. Supposing that all the possible configurations are equally probable (except for constraints) at the micro-level, the probability of finding a certain distribution at the aggregate level will be proportional to the number of possible configurations at the micro-level which produce the given distribution at the macro-level.

Let T_j be the number of consumers who choose alternative j.

The vector $T = [T_1, \ldots, T_n]$ therefore represents the choice distribution at the aggregate level. Obviously we have the constraint:

$$\sum_{j=1}^{n} T_j = P . \tag{11}$$

On the basis of the assumptions made above, the probability of observing the vector T is proportional to:

$$W(T) = \frac{P!}{\prod_{j=1}^{n} T_j!} . \tag{12}$$

The additional assumption that P and T_j, j = 1,. . .,n, are large allows us to use the Stirling approximation:

$$\ln T_j! \sim T_j (\ln T_j - 1) \tag{13}$$

Therefore, if we want to find the most probable distribution T, that is the one which maximises W(T), for large numbers it will be, with a good approximation, the distribution which maximises the quantity

$$\ln W(T) = - \sum_j \ln T_j! + \ln P!$$

since the search for the maximum is not affected by an increasing monotonic transformation of the maximising functions. Ignoring the constant ln P! which does not affect the location of the maximum and using the approximation (13) it follows that the most probable distribution T is that which maximises the function:

$$E(T) = - \sum_j T_j (\ln T_j - 1). \tag{14}$$

This function is known as the entropy of the distribution T.

According to Wilson and in analogy with the use of this method in statistical mechanics, the search for the maximum of E(T) is subject not only to constraint (11) but also to a further constraint of conservation of some kind of "total energy" of the system.

Wilson identifies this energy with total journey cost (or time). In the formulation used here, this is translated and generalised into the total utility of the system, that is the quantity:

$$\sum_j T_j v_j . \tag{15}$$

From the above, it follows that the most probable distribution T is the solution to the mathematical programming problem:

$$\max_T \; \{E(T): \sum_{j=1}^{n} T_j = P, \; \sum_{j=1}^{n} T_j v_j = V\} , \tag{16}$$

in which V is the given level of total utility. It is easily verified that the function E(T) is concave, therefore problem (16) is a concave programming problem (being subject to linear contraints). This means that the classic Lagrange multiplier method can be used to determine the unique solution.

The Lagrange formulation corresponding to problem (16) is:

$$\mathcal{L}(T,\nu,\mu) = E(T) - \nu\ (P-\sum_{j=1}^{n} T_j) - \mu\ (V-\sum_{j=1}^{n} T_j\ v_j), \tag{17}$$

in which ν and μ are the multipliers associated with constraints (11) and (15) respectively.

Eliminating the $\mathcal{L}$ derivatives with respect to each T_j we obtain:

$$\ln T_j = -(\nu + \mu\ v_j)\ ,$$

that is:

$$T_j = k\ e^{-\mu v_j} \tag{18}$$

in which

$$k = e^{-\nu}\ .$$

Constraint (11) allows us to eliminate the constant k from (18). In addition, without losing generality, we can define

$$\beta = -\mu$$

(in fact β is empirically always non-negative). We therefore have:

$$T_j = P\ \frac{e^{\beta v_j}}{\sum_j e^{\beta v_j}}\ , \tag{19}$$

which is the formula of the multinomial logit model in its simplest form. The value of the function E(T) associated with distribution (19) is given by:

$$E(T) = - \sum_j T_j \left(\beta v_j + \ln P - \ln \sum_j e^{\beta v_j} - 1\right) =$$

$$= P \ln \sum_j e^{\beta v_j} + \text{constants}$$

and as entropy is generally defined up to an additive arbitrary constant we can redefine it so that:

$$E(T) = P \ln \sum_j e^{\beta v_j} . \tag{20}$$

The entropy-maximising principle, having been borrowed from statistical mechanics, does not have a direct macro-economic interpretation. Despite this, economic behaviour is induced in the solution (19), which shows a tendancy for choices to be concentrated on the alternatives with higher utility.

This tendancy, obtained as a result by Wilson, is used by Smith (1978a,1983) as a starting assumption for his cost efficiency principle (which, given the formulation used here, could be more appropriately renamed utility efficiency principle).

Here we briefly examine how the simple problem of distribution is dealt with using the approach of Smith (1978a).

We consider once again a total population P, a set of alternatives $R = \{j : j = 1, \ldots, n\}$, and a utility v_j associated with each alternative $j \in R$.

We denote with:

$$t = (r_1, \ldots, r_P) \quad ; \quad r_k \in R,\ k = 1, \ldots, P,$$

a choice pattern consisting of the list of alternatives chosen by the first, second, n^{th} individual. For different realisations (for example, different days) we find in general different patterns t.

Let $T = (t_1, \ldots, t_m)$ be a sequence of realisations of different patterns and Q(T) the probability of observing T. It is shown by Smith (1982) that the form of Q(T) is completely determined by the following two assumptions, which constitute the essence of the utility efficiency method.

Assumption 1 (Independence)

The terms of the sequence T are independent. From this it follows:

$$Q(T) = \prod_{k=1}^{m} Q(t_k).$$

Assumption 2 (Utility efficiency principle)

Define:

$$\overline{V}(T) = \frac{1}{m} \sum_{k=1}^{m} V(t_k) \qquad \text{the average utility of the sequence T}$$

in which:

$$V(t) = \sum_{j=1} v(r_j) \tag{21}$$

is the cumulate utility for all P customers of the pattern $t = (r_1, \ldots, r_P)$.

If for two sequences T, T'

$$\overline{V}(T) \geq \overline{V}(T'),$$

then

$$Q(T) \geq Q(T').$$

Assumption 2 introduces an explicit condition of macro-economic regularity, which is however very weak (much weaker, for example, than utility maximisation). It simply says that it is more probable to observe sequences of choice patterns with high average utility than vice versa.

Smith shows that from assumptions 1 and 2 we can derive

$$Q(t) = k\, e^{\beta V(t)}, \qquad \beta \geq 0, \tag{22}$$

in which k is a normalising constant and β is a parameter.

The result (22) and the definition (21) are sufficient to show the equivalence between the utility efficiency principle and the entropy maximising principle.

In fact, if we let $F = (F_1, \ldots, F_n)$ be the vector of the distribution of the P consumers among the alternatives 1, . . ,n, such that:

$$\sum_{j=1}^{n} F_j = P \tag{23}$$

and we consider all the choice patterns t which produce the same vector F, the total number of patterns is given by:

$$\frac{P!}{\prod_{j=1}^{n} F_j!} \tag{24}$$

In addition, they all have the same cumulate utility:

$$V(t) = V(F) = \sum_{j=1}^{n} F_j v_j$$

and therefore, are equally probable, with a probability given by (22):

$$k \exp(\beta \sum_{j=1}^{n} F_j v_j) . \tag{25}$$

Combining (24) and (25) we find that the probability associated with a distribution F is proportional to:

$$W(F) = \frac{P!}{\prod_{j=1}^{n} F_j} \exp(\beta \sum_{j=1}^{n} F_j v_j) \tag{26}$$

and the most probable distribution F can be determined as the solution of the mathematical programming problems:

$$\max_F \{ \ln W(F) : \sum_{j=1}^{n} F_j = P \} . \tag{27}$$

Using as usual the assumption that the F_j are sufficiently large and Stirling's approximation

$$\ln F_j! \sim F_j(\ln F_j - 1),$$

we have:

$$\ln W(F) = \ln P! + \beta \sum_{j=1}^{n} F_j v_j - \sum_{j=1}^{n} \ln F_j! \sim$$

$$\sim - \sum_{j=1}^{n} F_j (\ln F_j - 1) + \beta \sum_{j=1}^{n} F_j v_j + \ln P!$$

Going back to (14) (definition of the entropy function) and ignoring the additive constants, we see that the problem (27) is equivalent to the problem

$$\max_F \{E(F) + \beta \sum_{j=1}^{n} F_j v_j \quad : \quad \sum_{j=1}^{n} F_j = P\}. \tag{28}$$

It is evident that problem (28) is a Lagrangian relaxation of problem (16) since a constraint of the type:

$$\sum_{j=1}^{n} F_j v_j = V$$

which appears in problem (16) is replaced by the term

$$\beta \sum_{j=1}^{n} F_j v_j$$

added to the objective function.

Therefore if the numerical value of the parameter β used in (28) is the same as that of the Lagrange multipliers β relative to the second constraint of (16), problems (16) and (28) are completely equivalent, that is, they have the same optimal solution and the objective function has the same optimum value. From this equivalence and from (19) it follows that the most probable distribution F based on utility efficiency is:

$$F_j = P \frac{e^{\beta v_j}}{\sum_j e^{\beta v_j}} \quad . \tag{29}$$

3.1.3. Random utility theory and entropy-maximisation

In 3.1.2. the equivalence between two macroscopic principles, one from statistical mechanics (entropy maximisation) and the other macro-economic (utility efficiency) was shown. In both cases it involved principles based on relatively weak assumptions and which imposed relatively few constraints on individual behaviours. In this section we examine a theory based explicitly on microscopic assumptions, i.e. random utility theory, and determine under what conditions it produces results equivalent to the macroscopic theories mentioned above.

We consider once again the choice situation discussed in 3.1.2. but examine in particular the behaviour of a single individual when facing the alternatives j, j = 1,. . . ,n.

In the aggregate approaches previously discussed we introduced a set of numerical weights v_j, called "utility" (even though neither of the two theories in strictly "utilitarian"), which measure the different relative attractiveness of the alternatives.

From a traditional utilitarian point of view, if the v_j were actually interpreted as utility functions defined by each member of the population P on the set of alternatives, and if the population were be composed of P perfectly homogeneous members with respect to their evaluation of the alternatives (i.e. with the same utility function), then an individual drawn at random from the P individuals would choose the alternative k for which the utility v_k is maximum:

$$v_k = \max_j \; v_j \quad . \tag{30}$$

Assumption (30), which will be defined as the deterministic utility maximisation, characterises most of the neoclassic urban economics models.

The principle of the choice of the alternative with maximum utility or minimum cost also constitutes the base of the classic theory of location of economic activities and services, from Weber's model to the assignment and location models proposed by Operational Research. (Some of the principal models of this type are described and discussed by Beckmann and Colorni in this book).

However, we know that this principle does not produce very realistic choice patterns. If the population P were really homogeneous, we would observe that the totality of choices would be concentrated on alternative k which corresponds to the maximum utility v_k, while all the other alternatives $j \neq k$ for which $v_j < v_k$, would be neglected.

A way of eliminating this undesirable aspect is to assume that the population P is heterogeneous with respect to the evaluation of alternatives, that is that each individual has a different utility function. As direct observation of the utility function of each single individual is impossible, a description in deterministic terms of a disaggregated choice process with a heterogeneous population is also impossible.

The theory of random utilities, proposed by Luce (1959) and developed by Manski (1973), Domencich and McFadden (1975), and Ben-Akiva and Lerman (1979) tackles this problem by explicitly introducing stochastic elements in the choice process. To solve the problem the theory suggests giving the following description in probability terms.

Suppose an individual chosen randomly from population P assigns to alternative j the utility

$$\tilde{u}_j = v_j + \tilde{\theta}_j, \qquad (31)$$

in which v_j are the quantities used previously, which can be interpreted as deterministic components of utility, identical for all individuals (and dependent on observable characteristics of alternative j), while $\tilde{\theta}_j$ are random variables which can be interpreted as stochastic components of utility, varying from individual to individual and independent of observable characteristics of the alternatives.

A simplifying assumption often introduced is that $\{\tilde{\theta}_j\}$ is a sequence of identically distributed independent random variables, for which a probability distribution is assumed to exist:

$$F(x) = P_r\{\tilde{\theta}_j \leq x\}, \quad j = 1, \ldots, n. \tag{32}$$

Assumptions (30) and (31) make it possible to obtain closed-form expressions for the distribution of maximum utility and the choice probabilities among alternatives.

First from (30) and (31) it follows that the probability distribution of total utility for alternative j is given by:

$$\begin{aligned} P_r\{\tilde{u}_j \leq x\} &= P_r\{v_j + \tilde{\theta}_j \leq x\} = P_r\{\theta_j \leq x - v_j\} = \\ &= F(x - v_j). \end{aligned} \tag{33}$$

Therefore defining the random variable:

$$\tilde{u} = \max_j \tilde{u}_j$$

and noting the equivalence between the two events:

$$\tilde{u} \leq x$$

and

$$\tilde{u}_j \leq x \qquad \text{for all } j = 1, \ldots, n,$$

it follows that:

$$\begin{aligned} H(x) &= P_r\{\tilde{u} \leq x\} = P_r\{\tilde{u}_1 \leq x, \ldots, \tilde{u}_n \leq x\} = \\ &= \prod_{j=1}^{n} P_r\{\tilde{u}_j \leq x\} = \prod_{j=1}^{n} F(x - v_j). \end{aligned} \tag{34}$$

The probability density h(x) associated with H(x) is:

$$h(x) = H'(x) = \sum_{j=1}^{n} F'(x - v_j) \prod_{k \neq j} F(x - v_k) \tag{35}$$

and the expected utility associated with the choice of an alternative with maximum total utility is:

$$V = \int_{-\infty}^{\infty} x \, d\, H(x) = \int_{-\infty}^{\infty} x \, h(x) \, dx =$$

$$= \int_{-\infty}^{\infty} x \sum_{j=1}^{n} F'(x - v_j) \prod_{k \neq j} F(x - v_k) \, dx \quad . \tag{36}$$

Finally, the choice probability for alternative j, that is, the probability of the event:

$$\tilde{u}_j = \max_k \tilde{u}_k$$

is given by

$$P_j = \int_{-\infty}^{\infty} F'(x - v_j) \prod_{k \neq j} F(x - v_k) \, dx, \tag{37}$$

and from (35) it follows that:

$$\sum_{j=1}^{n} P_j = \int_{-\infty}^{\infty} h(x) \, dx = 1 .$$

If specific assumptions are introduced on the form of the distribution F(x) of the random components of utility, the equations (34) - (37) assume different explicit forms. Particular importance has been given in the theory and the applications to the assumption:

$$F(x) = \exp\,(-e^{-\beta x}) \quad . \tag{38}$$

The distribution (38) is known as the extreme value distribution or Gumbel's distribution. Its importance is due to the fact that it implies the multinomial logit model, obtained already in (19) and (29). This can be shown more easily by introducting the transformation:

$$\tilde{y}_j = e^{-\beta \tilde{u}_j} = e^{-\beta(\tilde{\theta}_j + v_j)} \quad . \tag{39}$$

The sequence of random variables $\{\tilde{y}_j\}$ is a non increasing monotone transformation of the sequence of total utilities $\{\tilde{u}_j\}$, therefore $\tilde{y}_j$ can be considered a measure of the disutility of alternative j = 1,. . . ,n. It is easily seen that the random variables $\tilde{y}_j$ are distributed exponentially. In fact, if we let:

$$V_j = e^{\beta v_j}, \tag{40}$$

from (38) we obtain:

$$G_j(x) = P_r\{e^{-\beta(\tilde{\theta}_j + v_j)} \leq x\} = P_r\{\tilde{\theta}_j > -\frac{1}{\beta}\ln x - v_j\} =$$

$$= 1 - F(-\frac{1}{\beta}\ln x - v_j) = 1 - e^{-V_j x}. \tag{41}$$

Alternative j is chosen if its disutility $\tilde{y}_j$ is minimum, which occurs with the probability:

$$P_j = \int_0^{\infty} G'_j(x) \prod_{k \neq j} [1 - G_k(x)]\, dx =$$

$$= V_j \int_0^{\infty} e^{-x \sum_k V_k} dx = \frac{V_j}{\sum_k V_k}$$

or, substituting from (40):

$$P_j = \frac{e^{\beta v_j}}{\sum_k e^{\beta v_k}}. \tag{42}$$

Equation (42) is clearly identical to (19) and (29). The distribution of the random variable:

$$\tilde{y} = \min e^{-\beta \tilde{u}_j} = e^{-\beta \tilde{u}}$$

is given by:

$$L(x) = 1 - \prod_{j=1}^{n} [1 - G_j(x)] = 1 - e^{-\Phi x}, \tag{43}$$

having put

$$\Phi = \sum_j V_j = \sum_j e^{\beta v_j}.$$

The mean value of $\tilde{u}$, expressed as a function of $\tilde{y}$ by the equation:

$$\tilde{u} = -\frac{1}{\beta} \ln \tilde{y}$$

is therefore given by:

$$V = E\{\tilde{u}\} = E\{-\frac{1}{\beta} \ln \tilde{y}\} =$$

$$= -\frac{1}{\beta} \int_0^{\infty} \Phi e^{-\Phi x} \ln x \, dx =$$

$$= \frac{1}{\beta} \int_0^{\infty} (\ln \Phi - \ln y) \; e^{-y} \, dy =$$

$$= \frac{1}{\beta} \ln \Phi + \frac{\gamma}{\beta}, \tag{44}$$

in which γ is Eulero's constant.

It is important to note that the expected utility given by (44), apart from the additive constant γ/β which can be ignored, and the multiplicative $\frac{1}{\beta}$, is identical to the second term on the right-hand side of (20). In fact, the term

$$V = \frac{1}{\beta} \ln \Phi = \frac{1}{\beta} \ln \sum_j e^{\beta v_j}$$

is the expected utility of an optimal choice for a single individual. The expected utility aggregated for a population of P individuals is therefore:

$$PV = \frac{1}{\beta} P \ln \sum_j e^{\beta v_j} \tag{45}$$

and the comparison of (45) with (20) shows that the aggregated utility, as it is obtained from the theory of random utility under the assumption (38), is formally identical (except for a constant of proportionality) to the maximum value of the entropy function in the optimisation problem (16).

This analogy suggests a micro-economic interpretation of entropy, as well as a substantial equivalence between the theory of random utility and theory of entropy maximising (and, as a consequence, utility efficiency).

There is however one aspect of this reasoning which at first sight seems insatisfactory. While both entropy-maximisation and utility efficiency are principles based on very weak and not highly constraining assumptions at the level of individual behaviour, the results (42) and (45) were obtained using a very specific assumption on the heterogeneity of preferences, i.e. the distribution (38).

It is not immediately evident that (38) has general theoretical foundations and is not simply an ad hoc assumption made for the sake of the simplicity of the calculation.

In fact it is possible to derive (38) as an asyntotic result from weaker assumptions, exploiting the properties of maxima of sequences of random variables. This approach, apparently ignored in the literature, that implicitly considers assumption (38) as necessary to the derivation of the logit model (see, for example, Domencich and McFadden, 1975), has been explored and developed in some recent papers by Leonardi (1982b, 1982c).

In order to clarify this argument, let us assume that the alternatives are partitioned into n homogeneous classes 1,2,. . . ,j,. . . ,n, and define:

v_j the deterministic utility associated with an alternative from class j.

It is assumed in addition that each individual, in order to choose, considers a sample of alternatives drawn sequentially, and define:

w_j the probability that for a given draw, the alternative is drawn from class j, $w_j \geqslant 0$, $\sum_{j=1}^{n} w_j = 1$.

If $F(x) = P_r \{ \tilde{\theta}_j \leqslant x \}$ is the probability distribution for the random utilities, which for the moment we shall consider generic, it emerges that the utility distribution associated with any alternative at any draw is:

$$G(x) = \sum_j w_j \, F(x-v_j) \tag{46}$$

because of (33). The maximum utility distribution in a sample of N alternatives is therefore:

$$H_N(x) = G^N(x) \tag{47}$$

which is valid for any distribution F(x). We now introduce the following assumption concerning F(x):

$$\lim_{y\to\infty} \frac{1-F(x+y)}{1-F(y)} = e^{-\beta x}, \quad \beta > 0. \tag{48}$$

It is easily verified that the property (48) is equivalent to:

$$\lim_{x\to\infty} \frac{F'(x)}{1-F(x)} = \beta \ , \ \beta > 0. \tag{49}$$

(48) or (49) characterise a family of probability distributions for which the following theorem holds:

Theorem 1 (Leonardi)

Under the assumption (48) we have:

$$\lim_{N\to\infty} G^N \left(x + \frac{1}{\beta} \ln \Phi + a_N\right) = \exp\left(-e^{-\beta x}\right), \tag{50}$$

in which:

$$\Phi = \sum_{j=1}^{n} w_j \, e^{\beta v_j} \tag{51}$$

and a_N is the root of the equation:

$$1-F(a_N) = 1/N. \tag{52}$$

Proof of theorem 1

From (52) it follows that:

$$a_N = F^{-1}\left(1-\frac{1}{N}\right)$$

and therefore:

$$\lim_{N\to\infty} a_N = F^{-1}(1) = \infty.$$

Hence, from the property (48) it follows that:

$$\lim_{N\to\infty} \frac{1-F\left(x-v_j+\frac{1}{\beta}\ln \Phi + a_N\right)}{1-F(a_N)} = \frac{e^{\beta v_j}}{\Phi} e^{-\beta x},$$

or:

$$\lim_{N\to\infty} F\left(x-v_j+\frac{1}{\beta}\ln \Phi + a_N\right) = \lim_{N\to\infty} \left\{1 - \frac{e^{\beta v_j}}{\Phi} e^{-\beta x} \left[1-F(a_N)\right]\right\}$$

and substituting from (52):

$$\lim_{N\to\infty} F(x-v_j + \frac{1}{\beta}\ln\Phi + a_N) = \lim_{N\to\infty} (1 - \frac{e^{\beta v_j}}{N\Phi} e^{-\beta x}) . \tag{53}$$

Substituting the result (53) in (46), we obtain:

$$\lim_{N\to\infty} G(x+\frac{1}{\beta}\ln\Phi + a_N) = \lim_{N\to\infty} (1 - \frac{e^{-\beta x}}{N\Phi} \sum_{j=1}^{n} w_j e^{\beta v_j})$$

and from the definition (51):

$$\lim_{N\to\infty} G(x+\frac{1}{\beta}\ln\Phi + a_N) = \lim_{N\to\infty} (1 - \frac{e^{-\beta x}}{N}) .$$

In conclusion:

$$\lim_{N\to\infty} G^N (x+\frac{1}{\beta}\ln\Phi + a_N) = \lim_{N\to\infty} (1 - \frac{e^{-\beta x}}{N})^N =$$

$$= \exp(-e^{-\beta x})$$

which establishes result (50). *Q.E.D.*

Intuitively, theorem 1 states that, if we can suppose that each individual considers for the choice a sufficiently large sample of alternatives of total size N the distribution of the utility associated with the best alternative is approximately:

$$H_N(x) \sim \exp[-\Phi e^{-\beta(x-a_N)}] \tag{54}$$

and the bigger N the better this approximation is. The constant a_N has only the effect of shifting the origin of the utility scale and therefore does not influence the choice behaviour. Except for additive constants, the mean of the distribution (54) is:

$$V = \frac{1}{\beta}\ln\Phi$$

which is in agreement with (44).

It is important to note that theorem 1, with its consequent results, was obtained without any specific assumptions about the form of the distribution F(x). Instead a weak property, (48), is assumed, which characterizes a wide family of distributions. (48) is in fact satisfied by most of the more well-known distributions and intuitively requires that the tail of the distribution F(x) can be asyntotically approximated with an exponential.

In any case, from the point of view of the constraints posed on individual behaviour, assumption (48) is of a level of generality which can be compared with that of the principles of entropy maximising on utility efficiency.

As a complement to theorem 1, which establishes the asyntotic form of the utility distribution, we give an analagous result on the asyntotic form of the choice probability. Define:

$p_j(N)$ the probability that an individual who has drawn a sample of size N (in the way described above) chooses an alternative of type j, $j = 1,\ldots,n$.

In this case the following theorem is valid,

Theorem 2 (Leonardi)

Under the same assumptions as for theorem 1:

$$\lim_{N\to\infty} p_j(N) = \frac{w_j\, e^{\beta v_j}}{\sum_{j=1}^{n} w_j\, e^{\beta v_j}} \quad . \tag{55}$$

Proof of theorem 2.

For any N, the choice probability of alternative j is given by:

$$p_j(N) = N w_j \int_{-\infty}^{\infty} F'(x-v_j)\, G^{N-1}(x)\, dx \quad . \tag{56}$$

In fact if N-1 alternatives have a utility of less than x, which occurs with probability $G^{N-1}(x)$, and the remaining alternative is of type j and has utility in (x, x + dx), which occurs with a probability $w_j F'(x-v_j)dx$, this last alternative is the best. This can occur in N different ways and for all values of x, hence (56).

We can define the following function of $v_1,\ldots,v_n$:

$$V_N(v_1,\ldots,v_n) = \int_{-\infty}^{\infty} x\, d\, G^N(x)\, , \tag{57}$$

which is none other than the expected utility associated with the alternative with maximum utility in a sample of size N.

It is easily observed that:

$$N\, w_j\, F'(x-v_j)\, G^{N-1}(x) = -\frac{\partial G^N(x)}{\partial v_j} \tag{58}$$

and therefore:

$$p_j(N) = -\int_{-\infty}^{\infty} \frac{\partial G^N(x)}{\partial v_j}\, dx =$$

$$= -x \left.\frac{\partial G^N(x)}{\partial v_j}\right|_{-\infty}^{+\infty} + \int_{-\infty}^{\infty} x\, d\, \frac{\partial G^N(x)}{\partial v_j} \,. \tag{59}$$

From (58) it follows that the first term on the right-hand side of (59) is zero, as:

$$F'(\infty) = 0 \quad e \quad G(-\infty) = 0 \quad .$$

Therefore the definition (57) and (59) imply the property

$$p_j(N) = \frac{\partial}{\partial v_j} V_N(v_1,\ldots,v_n) \quad . \tag{60}$$

That is, the choice probabilities are the derivatives of total expected utility with respect to the deterministic utilities. This property is valid for any N including $N\to\infty$.

Since:

$$\lim_{N\to\infty} V_N = V = \frac{1}{\beta} \ln \Phi = \frac{1}{\beta} \ln \sum_j e^{\beta v_j} w_j$$

as already seen, (up to an additive constant, independent of $v_1,\ldots,v_n$) we have:

$$\lim_{N\to\infty} p_j(N) = \frac{\partial V}{\partial v_j} = \frac{w_j e^{\beta v_j}}{\sum_j w_j e^{\beta v_j}}$$

which establishes the result (55). *Q.E.D.*

Theorem 2 suggests a complete correspondence between random utility theory, entropy maximising and also utility efficiency theory, as implied by the equivalence proved by Smith (in this book).

It should be noted that (55) is slightly more general than (19), in that each alternative is weighted differently with weights w_j. This however does not constitute a real structural difference between the two models. If we assume that the w_j are proportional to the number of alternatives of type j, that is:

$$w_j = k\,N_j\,,$$

in which

N_j is the total number of alternatives of type j.

Equation (19), applied to such a situation and counting the alternatives correctly would give:

$$T_j = P\,\frac{N_j\,e^{\beta v_j}}{\sum_j N_j\,e^{\beta v_j}} = P\,\frac{w_j\,e^{\beta v_j}}{\sum_j w_j\,e^{\beta v_j}}\,.$$

We can now establish rigorously the correspondence between random utility theory and entropy maximisation.

We define the function in $\mathbb{R}^n$

$$F(v) = \frac{1}{\beta}\ \ln \sum_{j=1}^{n} e^{\beta v_j}\,w_j\,, \tag{61}$$

which, as we know, is the expected utility associated with an optimal choice.

F(v) is a convex function, therefore it is possible to define its conjugate function in terms of the Legendre transform:

$$\mathcal{L}(y) = \max_{v}\ \{\sum_j v_j\,y_j - F(v)\}\ . \tag{62}$$

The vanishing of the derivitives of the right-hand side yields:

$$y_j = \frac{w_j\,e^{\beta v_j}}{\sum_j w_j\,e^{\beta v_j}} \tag{63}$$

and, as we know, these are the choice probabilities. The Legendre transform thus establishes a duality between the space of vector v, which are the deterministic utilities, and the space of vectors y, which are the choice probabilities. From (63) we have:

$$v_j = \frac{1}{\beta}\left(\ln \frac{y_j}{w_j} + \ln \Phi\right), \quad \Phi = \sum_j w_j e^{\beta v_j}$$

substituting this result in (62) we obtain:

$$\mathcal{L}(y) = \frac{1}{\beta} \sum_j y_j \ln \frac{y_j}{w_j} + \frac{1}{\beta} \ln \Phi - \frac{1}{\beta} \ln \Phi =$$

$$= \frac{1}{\beta} \sum_j y_j \ln \frac{y_j}{w_j} \quad . \tag{64}$$

The function:

$$E(y) = \frac{1}{\beta} \sum_j y_j \ln \frac{y_j}{w_j} \tag{65}$$

is clearly an entropy (changed in sign). It is thus established that entropy is the Legendre transform of the expected utility.

The Legendre transformation is reversible, that is it is possible to determine the conjugate function of the function E(y) which is also convex:

$$\mathcal{L}^*(v) = \max_y \{\sum_j y_j v_j - \frac{1}{\beta} \sum_j y_j \ln \frac{y_j}{w_j}\}. \tag{66}$$

Since the space of the vectors y is normalised, the maximisation in (66) is subject to the constraint:

$$\sum_j y_j = 1. \tag{67}$$

Introducing a Lagrange multiplier ν for constraint (67) and equating the derivatives of the right-hand terms of (66) to it, we obtain:

$$v_j - \frac{1}{\beta}\left(\ln \frac{y_j}{w_j} + 1\right) = \nu,$$

hence:

$$y_j = k\, w_j\, e^{\beta v_j}, \quad k = e^{-(1+\beta\nu)}$$

Finally, eliminating the constant k by means of constraint (67) we have:

$$y_j = \frac{w_j e^{\beta v_j}}{\sum_j w_j e^{\beta v_j}}, \tag{68}$$

which is identical to (63).

Defining:

$$\Phi = \sum_j w_j e^{\beta v_j}$$

and substituting (68) in (66), we obtain:

$$\mathcal{L}^*(v) = \sum_j v_j y_j - \sum_j y_j v_j + \frac{1}{\beta} \ln \Phi =$$

$$= \frac{1}{\beta} \ln \Phi = \frac{1}{\beta} \ln \sum_j w_j e^{\beta v_j} = F(v). \tag{69}$$

That is, the Legendre transform of the entropy function E(y) is the expected utility F(v).

We have thus established the following theorem:

Theorem 3 (Leonardi).

The function F(v) (expected utility) and E(y) (entropy) defined above are reciprocally conjugate in the duality induced by the Legendre transform. The two spaces placed in duality are the space of deterministic utilities v,

$$v \in \mathbb{R}^n,$$

and the space of the choice probabilities y,

$$y \in S,$$

in which

$$S = \{ x : x \in \mathbb{R}^n, \sum_j x_j = 1 \}.$$

The result of theorem 3 is valid for the behaviour of a single individual. It can however be extended without difficulty to the case of P individuals. In this case the choice probability will be replaced by expected flows, that is, by the distribution of the P individuals among the various alternatives, simply by multiplying P by y. We find in this way that the choice behaviour has two completely equivalent alternative representations. One representation is in the utility space, through the expected utility function F(v), corresponding to random utility theory, and the other representation is in the flow space, through the entropy function E(y) corresponding to the principle of entropy maximising.

3.1.4. Conclusions

The equivalence between the three different approaches to modelling of the choice behaviour with probabilistic dispersion was shown with the use of a simple model, which can be considered the basic prototype of all choice models in which individual behaviour is independent of that of other individuals. More complex cases can be imagined in which factors such as a limited number of alternatives or limited capacity or excessive congestion introduce effects of mutual disturbance, inhibition and competition among individuals. In such cases the choices are subject to further constraints and in general the utility of each alternative depends on the number of individuals who choose it. In other words, it is necessary to introduce endogenous signals of scarcity, in the form of negative externalities or prices.

A comparative analysis of the different approaches to the introduction of such signals is the subject of a later section.

A final comment concerns the relationship between random utility maximising and deterministic utility maximising. It is obvious that the latter can be deduced as a limiting case of the former, when the assumption of heterogeneity of the population is dropped.

More precisely, it has been shown by Evans (1973) that a model of type (19), (29), or (55) coincides with the deterministic choice model when the parameter β goes to infinity. As, according to the interpretation of the theory of random utilities, β is inversely proportional to the variance of the distribution (38), this limiting case coincides with that in which the distribution of the random utility terms degenerates into a concentration on a single value, that is, the heterogeneity of the population disappears.

This can be demonstrated even without assuming a specific form for the distribution (38). We suppose that the distribution F(x) degenerates into a concentration on a given value a, that is:

$$F(x) = \begin{cases} 0, & x \leq a \\ 1, & x > a \end{cases} \tag{70}$$

In this case, the corresponding density will be:

$$F'(x) = \delta(x-a) \tag{71}$$

where $\delta(x\text{-}a)$ is a Dirac delta lumped on the value a.

Substituting (70) and (71) in (37) and noting that:

$$\prod_{k\neq j} F(x-v_k-a) = \begin{cases} 0, & x \leq \max\limits_{k\neq j}(v_k+a) \\ 1, & x > \max\limits_{k\neq j}(v_k+a) \end{cases}$$

we obtain:

$$P_j = \int_{\max\limits_{k\neq j}(v_k+a)}^{\infty} \delta(x-v_j-a)\,dx = \begin{cases} 1, \text{ if } & v_j > \max\limits_{k\neq j} v_k \\ 0, \text{ if } & v_j < \max\limits_{k\neq j} v_k \end{cases} \tag{72}$$

That is, the choice concentrates on the alternative with maximum utility, which is precisely the neo-classic principle of deterministic utility maximisation.

3.2. *Mechanism of price formation and spatial differentiation in location-transport systems*

3.2.1. Introduction

Any system of transport and location will always involve the interaction between demand flows on the one hand and stocks of goods and services on the other. The former are mobile and subject to relatively fast changes, the latter are immobile and subject to very slow changes.

This difference inevitably creates situations of disequilibrium and gives rise to self-regulating mechanism which try to increase or reduce the demand flows depending upon whether stock is abundant or scarce. These mechanisms take the form of negative externalities, which we refer to as "prices". This concept of price includes both monetary and non-monetary prices. Housing rents for example are clearly expressed in money terms, but other important signals, such as queuing time for housing assignment (discussed by Weibull in this book), often occurring in publicly regulated housing markets, can also be considered as a price. In the same way, the time taken to travel a certain distance in a congested network (as discussed by Smith in this book) is also a price.

These prices do not always reflect the existence of an actual market. For this to be the case it is necessary for the stock of goods and services offered to be controlled by suppliers who will attempt in some way to make a profit from their sale. This is typically the case of rents and house prices and also retail prices in shopping centres. It it not true however of journey times for example, which are purely physical phenomena determined by traffic flow and levels of congestion.

Although the definition of price used here is essentially dynamic, as prices are seen as a mechanism of adaptation to disequilibrium, most classic theory concerning price setting in spatial and multi-regional systems is based on situations of static equilibrium. It is clear however that a dynamic formulation constitutes the most natural future development for a spatial theory of prices.

A further important distinction concerns the assumption made about the information possessed by consumers and suppliers. In classic theory it is assumed that both parties have perfect knowledge of the market and that demand is homogeneous and non-stochastic. In certain more recent studies although demand is considered to be heterogeneous and stochastic, the supplier is still assumed to have perfect knowledge of demand (cf.: Anas, 1979, 1982). Leonardi in this book however makes no a priori assumption about knowledge on the part of the supplier of the relationship between price and demand. He considers prices to be generated in real time through direct bargaining between consumer and supplier.

We now analyse if and to what extent the progressive introduction of stochastic elements in the classic deterministic static approach allows us to construct a coherent general frame of reference.

3.2.2. The deterministic equilibrium model and linear programming

We consider once again a situation in which the consumer has to choose between a discrete set of alternatives. Each consumer can choose only one alternative. From the supply-side each alternative is controlled by an entrepreneur who sets the price.

We define:

i,j the subscript associated with different zones $i,j = 1,\ldots,m$;

v_{ij} the utility of an alternative in zone j for a consumer in zone i (v_{ij} may also include the cost of transport between i and j), if the alternative is offered at zero price;

r_j the price of an alternative offered in zone j.

We suppose that the consumer seeks to maximise utility and that the supplier seeks to maximise profit. It is also assumed that the utility function is linear in prices.

This implies that if each consumer in zone i chooses in such a way as to maximise his utility and if market prices are r_j, $j = 1,\ldots,m$, in an equilibrium situation, the utility for a consumer will be:

$$V_i = \max_j (v_{ij} - r_j) \quad , \quad i = 1, \ldots, m \, . \tag{73}$$

If (73) is satisfied by a particular k, we have

$$v_{ik} - r_k = V_i \quad , \text{ if } v_{ik} - r_k = \max_j (v_{ij} - r_j) \, ,$$

or:

$$r_k = v_{ik} - V_i \tag{74}$$

As the suppliers maximise profit, the price will be set equal to the highest level consistent with (74), that is by choosing the consumer willing to pay the highest price. Hence,

$$r_j = \max_i \ (v_{ij} - V_i), \quad j = 1, \ldots, m. \tag{75}$$

Equations (73) and (75) constitute the equilibrium conditions of the market in the classic sense of the term. These conditions, even if derived through micro-economic reasoning, are equivalent to the optimum conditions of a linear programming problem. If we consider the problem:

$$\max_x \ \{\sum_{ij} x_{ij} v_{ij} : \ \sum_j x_{ij} = P_i, \quad \sum_i x_{ij} = Q_j, \quad x_{ij} > o\} \tag{76}$$

in which:

P_i is the total number of consumers in zone i;

Q_j is the total supply (number of alternatives available) in zone j, and $\sum_j Q_j = \sum_i P_i$.

The dual problem associated with (76) is:

$$\min_{\nu, \mu} \ \{\sum_i P_i \nu_i + \sum_j Q_j \ \mu_j : \ \nu_i + \mu_j \geqslant v_{ij}\} , \tag{77}$$

in which ν_i, μ_j are the shadow prices associated with the constraints of origin and destination respectively. From the structure of the objective function and the constraints of (77) it follows that μ_j must satisfy the condition:

$$\mu_j = \max_i \ (v_{ij} - \nu_i), \ j = 1, \ \ldots, \ m \ . \tag{78}$$

Reasoning in the same way, ν_i must satisfy the condition:

$$\nu_i = \max_j \ (v_{ij} - \mu_j) \quad , \ i = 1, \ \ldots, \ m \ . \tag{79}$$

Comparing (78) with (75) and (79) with (73) we find that the shadow prices μ_j are identifiable with the real prices r_j while the shadow prices ν_i are identifiable with the utilities V_i.

Problem (76) is one of the simplest problems of linear programming (known as the "assignment problem") and can be considered as a basic prototype for a model of general equilibrium of a spatial market.

When the goods exchanged are dwellings, problem (76) is essentially the same as the model of Herbert and Stevens (1960), which constitutes a translation into discrete space of the classic theory of housing market equilibrium (Alonso, 1964a).

An important feature of the model expressed in (76) and (77) is its perfect consistency in neo-classic terms. The conditions (78) and (79), which as shown are equivalent to (73) and (75), imply that the "rational" behaviour of consumers and producers at the individual microscopic level (the maximisation of utility and maximisation of profit respectively) is equivalent to the rational behaviour of the system at macroscopic level. The objective function of the dual problem (77) can be broken down into two elements:

$\sum_i P_i \nu_i$ aggregated utility for all consumers or, in other words, consumer surplus,;

$\sum_i Q_j \mu_j$ aggregated profit for all producers or, in other words, producer surplus.

The sum of the two components is the so-called "Total Social Benefit". The conditions (78) and (79) plus the observations made above show that individual optimum behaviour of demand and supply implies the optimisation of Total Social Benefit.

This result is obtained assuming that the utility function is linear, that there is a state of general equilibrium, that demand is homogeneous and that information is perfect for consumers and producers.

We now show how the result can be generalised (or changed by various degrees) by relaxing one or more of these assumptions.

3.2.3. The stochastic equilibrium model and entropy-maximisation

A first step towards generalisation is to keep all the previous assumption, except the homogeneity of demand. We therefore suppose that consumers have different preferences and that this heterogeneity is known stochastically. As in random utility theory this can be modelled assuming that the utility of an alternative in j for a consumer in i is given by:

$$v_{ij} - r_j + \tilde{\varepsilon}_{ij} ,$$

in which v_{ij} and r_j are defined as previously and $\tilde{\varepsilon}_{ij}$ is a random variable which reflects the heterogeneity of preference of the consumers. The variables $\{\tilde{\varepsilon}_{ij}\}$ are assumed to be independent and identically distributed with a distribution:

$$P_r \{\tilde{\varepsilon}_{ij} \leqslant x\} = F(x) .$$

As an additional hypothesis, it is also assumed that F(x) is asyntotically exponential in its right-hand tail, that is, that the following property is valid:

$$\lim_{y\to\infty} \frac{1 - F(x + y)}{1 - F(y)} = e^{-\beta x} , \quad \beta > o . \tag{80}$$

The equations analogous to (73) and (75) are respectively:

$$V_i = \max_j \,(v_{ij} - r_j + \tilde{\varepsilon}_{ij}) \tag{81}$$

$$r_j = \max_i \,(v_{ij} - V_i + \tilde{\varepsilon}_{ij}) \tag{82}$$

Since $\tilde{\varepsilon}_{ij}$ are random variables, the utility V_i and the prices r_j are also random variables.

When the number of consumers (equal to the number of alternatives) is large, the distribution of V_i and r_j and also the resulting choice model, assume a particularly simple form. In fact we have the following:

Theorem 4 (Leonardi)

Let the property (80) hold, and $\sum_i P_i = \sum_j Q_j = Q$ and let there be constants $\omega_j > 0$, $\sum_j \omega_j = 1$, $\alpha_i > 0$, $\sum_i \alpha_i = 1$, such that

$$Q_j = Q\omega_j , \qquad P_i = Q \alpha_i . \tag{83}$$

Define also:

$$A(Q) = F^{-1}(1 - \frac{1}{Q}) . \tag{84}$$

Then,

$$\lim_{Q \to \infty} P_r \{V_i - A(Q) \leqslant x \mid r_j = y_j\} = \exp \{-\phi_i(y)\, e^{-\beta x}\} \tag{85}$$

$$\lim_{Q \to \infty} P_r \{r_j - A(Q) \leqslant x \mid V_i = y_i\} = \exp \{-\psi_j(y)\, e^{-\beta x}\}, \tag{86}$$

where:

$$\phi_i(y) = \sum_j e^{\beta(v_{ij} - y_j)}\, \omega_j \tag{87}$$

$$\psi_j(y) = \sum_i \alpha_i \cdot e^{\beta(v_{ij} - y_i)}. \tag{88}$$

Proof of theorem 4.

Equation (84) implies that:

$$\lim_{Q \to \infty} A(Q) = F^{-1}(1) = \infty \tag{89}$$

and also that:

$$1 - F[A(Q)] = 1/Q. \tag{90}$$

From the property (80) and from (89) and (90) it follows that:

$$\lim_{Q \to \infty} 1 - F[x - v_{ij} + y_j + A(Q)] = \lim_{Q \to \infty} \frac{1}{Q}\, e^{-\beta x}\, e^{\beta(v_{ij} - y_j)}. \tag{91}$$

From the definition (81) and the independence of the random utilities we obtain:

$$P_r \{V_i - A(Q) \leqslant x \mid r_j = y_j\} =$$

$$= P_r \{\max_{(j)} (v_{ij} - y_j + \tilde{\varepsilon}_{ij}) \leqslant x + A(Q)\} =$$

$$= \prod_j P_r \{v_{ij} - y_j + \varepsilon_{ij} \leqslant x + A(Q)\}^{\omega_j Q} =$$

$$= \prod_j F^{\omega_j Q} [x + A(Q) - v_{ij} + y_j] =$$

$$= \prod_j \{1 - [1 - F[x + A(Q) - v_{ij} + y_j]]\}^{\omega_j Q}.$$

Consequently, substituting from (91).

$$\lim_{Q\to\infty} P_r \{V_i - A(Q) \leqslant x \mid r_j = y_j\} =$$

$$= \lim_{Q\to\infty} \prod_j [1 - \frac{1}{Q} e^{-\beta x} e^{\beta(v_{ij} - y_j)}]^{\omega_j Q} =$$

$$= \prod_j \exp[-\omega_j e^{-\beta x} e^{\beta(v_{ij} - y_j)}] =$$

$$= \exp[-\Phi_i(y) e^{-\beta x}] ,$$

in which $\Phi_i(y)$ is defined by (87). The result (85) of theorem 4 is therefore proved. The result (86) can be proved in the same way. *Q.E.D.*.

From the above it follows that, apart from an additive constant A(Q), which simply has the effect of shifting the origin of the utility and prices and which is any case is arbitrary, both the utility distribution, which depends on prices, and the price distribution, which depends on utilities, are extreme values distributions (Galambos, 1978).

Their modes are respectively:

$$\nu_i = \frac{1}{\beta} \ln \phi_i(\mu) \tag{92}$$

$$\mu_j = \frac{1}{\beta} \ln \psi_j(\nu), \tag{93}$$

in which ϕ_i and ψ_i are defined from (87) and (88). The probability of a consumer choosing an alternative in j, is given, using (85), by the logit formula:

$$p_{ij} = \frac{e^{\beta(v_{ij} - \mu_j)} \omega_j}{\phi_i(\mu)} \tag{94}$$

Analogously, the probability that a supplier in j sells to a consumer in i, using (86), is given by the logit formula:

$$q_{ji} = \frac{e^{\beta(v_{ij} - \nu_i)} \alpha_i}{\psi_j(\nu)} . \tag{95}$$

It is easy to verify that demand and supply find an equilibrium for each (i, j) pair i.e. that:

$$P_i \, p_{ij} = Q_j \, q_{ji}. \tag{96}$$

In fact, substituting in (96) from (87) and (88) we obtain with one or two steps:

$$\frac{\psi_i}{\phi_i} = \frac{e^{\beta \mu_j}}{e^{\beta \nu_i}} \tag{97}$$

and this equality is obviously true because of (92) and (93).

It is also simple to show a precise relationship between (92) - (95) and a classic entropy-maximisation problem through.

Theorem 5.

The mathematical programming problem

$$\max_x \; \{\sum_{ij} x_{ij} \; (v_{ij} - \frac{1}{\beta} \log x_{ij}) \; : \; \sum_j x_{ij} = P_i, \; \sum_i x_{ij} = Q_j\} \tag{98}$$

has as primal solution:

$$x_{ij} = P_i \, p_{ij} = Q_j \, q_{ji}$$

and as dual variables ν_i and μ_j.

This result has already been proved and is well known (see Wilson, 1970a, for the derivation of this and analogous results).

The result which we are interested in here, but which is in general ignored, is the complete equivalence between the micro-economic formulation of theorem 4 and the entropic formulation of theorem 5.

We can show that the dual equation (98), up to additive constants, is given by:

$$D(\mu) = \frac{1}{\beta} \sum_i P_j \log \phi_i \, (\mu) + \sum_j Q_j \, \mu_j =$$

$$= \sum_i P_i \, \nu_i + \sum_j Q_j \, \mu_j \quad ,$$

a result which is essentially identical to (77) and can also be interpreted as the sum of consumer surplus and producer surplus.

3.2.4. Spatial differentiation of prices and utility

Given the stochastic formulation of the preceding point it is clear that prices and utilities will never be uniform as they are subject to random variations within the same zone. It is interesting to analyse whether apart from random fluctuations there are other systematic causes of the variations (or uniformity) of prices and utility and how such causes can be traced back to geographical factors. For this analysis it is sufficient to consider the modes (92) and (93). In order to make the transport in the following way:

$$v_{ij} = - c_{ij}$$

in which

c_{ij} is the cost of transport between i and j.

We therefore have:

$$\phi_i(\mu) = \sum_j e^{-\beta(c_{ij} + \mu_j)} \omega_j \tag{99}$$

$$\psi_j(\nu) = \sum_j \alpha_i e^{-\beta(c_{ij} + \nu_i)} . \tag{100}$$

Equation (99) can clearly be interpreted as a measure of accessibility from i to all the alternatives. The difference between (99) and a classic accessibility indicator (cf.: for example, Hansen, 1959) is that the total access cost to j from i also contains the destination price μ_j and not only the transport cost c_{ij}.

Similarly (100) can be interpreted as a demand potential in j from all the origins. Here too the total measure of distance contains an added term, the utility at the origin. This means that the attractiveness of j for consumers in i decreases as their utility ν_i decreases. In fact a high value of ν_i implies in general strong competition from other alternatives, that is, a vast range of choice for consumers in i and therefore a lesser probability that they choose alternative j.

From (99) and (100), and the geographical interpretation just given, it is clear that the only case in which accessibility and potential (and therefore utility and prices) are uniform is that in which there are no differences in transport costs. In fact if:

$$c_{ij} = c \quad , \quad \forall ij \, ,$$

then

$$\phi_i(\mu) = \sum_j e^{-\beta(c+\mu_j)} \omega_j = \phi(\mu) \quad , \forall i,$$

$$\psi_j(\nu) = \sum_i \alpha_i e^{-\beta(c+\nu_i)} = \psi(\nu), \quad \forall j$$

and therefore from (92) and (93):

$$\nu_i = \frac{1}{\beta} \ln \phi(\mu) = \nu, \quad \forall i$$

$$\mu_j = \frac{1}{\beta} \ln \psi(\nu) = \mu, \quad \forall j.$$

On the other hand, let:

$$\nu_i = \nu \quad , \forall i$$

$$\mu_j = \mu \quad , \forall j.$$

Then, because of (92) and (93):

$$\nu = \frac{1}{\beta} \ln \sum_j e^{-\beta c_{ij}} \omega_j - \mu$$

$$\mu = \frac{1}{\beta} \ln \sum_i \alpha_i e^{-\beta c_{ij}} - \nu \, ,$$

or

$$\nu + \mu = \frac{1}{\beta} \ln \sum_j e^{-\beta c_{ij}} \omega_j$$

$$\nu + \mu = \frac{1}{\beta} \ln \sum_i \alpha_i e^{-\beta c_{ij}} .$$

These equations hold only if:

$$\sum_j e^{-\beta c_{ij}} \omega_j = \sum_i \alpha_i e^{-\beta c_{ij}} . \qquad (101)$$

In general, the equality (101) is true only if:

$$c_{ij} = c , \quad \forall i,j .$$

3.2.5. General equilibrium with a given supply function

The cases considered up to now have been based on precise micro-economic assumptions on the behaviour of both demand and supply. In particular it is assumed that the supply-side will maximise profit subordinately to stock constraints and that this assumption is sufficient (added to that of utility maximisation for the consumer) to determine the equilibrium configuration of prices. We now look at the case in which the demand has the same behaviour as before, but the supply sets prices through a mechanism which is not necessarily, or at least not explicitly, bound to the maximisation of profits. This includes both the case of profit-maximising suppliers although with heterogeneities in information and decision rules which are unobservable at a disaggregate level, and the case where supply has no profit maximising management (and possibly no management at all).

In both cases the supply behaviour will be described not at a micro-economic level as previously, but directly at a macro-economic level postulating the existence of a set of supply functions such that:

μ_j (D_j) is the offered price for alternative j, when the demand level is D_j.

In this formulation constraints on the stock of available alternatives no longer appear explicitly. In theory, an unlimited number of consumers has access to each alternative. However, the supply price acts as a negative externality, which will alter the attractiveness of the alternative as a function of the number of consumers requesting it by inhibiting or incentivating the demand.

The fact that in theory the number of alternatives is unlimited and not simply very large makes the assumptions of theorem 4 artificial, as it is not plausible to suppose that each consumer has perfect knowledge of an infinite number of trials.

We can therefore replace the assumption of utility maximisation with the assumption of the choice of a "satisfycing" alternative. We suppose that each consumer collects information about a sequence of alternatives, randomly drawn from those available, until he finds one with a utility which reaches a certain satisfycing threshold level.

This change of assumptions is more apparent than real, as we shall show, since the choice of a satisfycing solution for high thresholds of utility is equivalent to utility maximisation with complete information.

However in this case the formulation in terms of a satisfycing choice is more natural as it introduces in a simple way the idea of a process of learning by trial and error and a rule for interrupting such a process.

The above mentioned result is stated rigorously in the following:

Theorem 6 (Leonardi)

Let property (80) hold, and assume that consumers have satisfycing behaviour with threshold utility y and let the prices μ_j, j = 1,...,m be given. Then the choice probabilities p_{ij} for a consumer in i of an alternative in j satisfy the asymptotic property:

$$\lim_{y\to\infty} p_{ij} = \frac{e^{\beta(v_{ij} - \mu_j)}}{\sum_j e^{\beta(v_{ij} - \mu_j)}} . \tag{102}$$

Proof of theorem 6

We introduce the notation:

$$F_{ij} = P_r \{v_{ij} - \mu_j + \tilde{\varepsilon}_{ij} \leqslant y\} = P_r \{\tilde{\varepsilon}_{ij} \leqslant y - v_{ij} + \mu_j\} = F(y - v_{ij} + \mu_j),$$

which is the probability that an alternative of type j is rejected.

A consumer in i can choose an alternative of type j in several mutually exclusive and exaustive ways. Proceeding inductively, he can draw a type j alternative at the first trial, with probability $\frac{1}{m}$ and accept it, with probability $1 - F_{ij}$. In general, after n-1 trials, which have been unsuccesful with probability $(\frac{1}{m} \sum_j F_{ij})^{n-1}$, the n^{th} trial is succesful and results in a j-type choice with probability $\frac{1}{m}(1-F_{ij})$. Letting $n \to \infty$ and using the theorem of total probability, we have:

$$P_{ij} = \frac{1}{m}(1-F_{ij}) + (\frac{1}{m}\sum_j F_{ij})\frac{1}{m}(1-F_{ij}) + (\frac{1}{m}\sum_j F_{ij})^2 \frac{1}{m}(1-F_{ij}) + \ldots =$$

$$= \frac{1}{m}(1-F_{ij}) \sum_{n=0}^{\infty} (\frac{1}{m}\sum_j F_{ij})^n = \frac{\frac{1}{m}(1-F_{ij})}{1-\frac{1}{m}\sum_j F_{ij}} = \frac{1-F_{ij}}{\sum_j (1-F_{ij})} .$$

On the other hand from the definition of F_{ij} and property (80) we have:

$$\lim_{y \to \infty} (1-F_{ij}) = \lim_{y \to \infty} [1-F(y)]\, e^{\beta(v_{ij} - \mu_j)} .$$

Hence, substituting in the previous result we obtain for P_{ij}:

$$\lim_{y \to \infty} P_{ij} = \lim_{y \to \infty} \frac{[1-F(y)]\, e^{\beta(v_{ij} - \mu_j)}}{[1-F(y)] \sum_j e^{\beta(v_{ij} - \mu_j)}} = \frac{e^{\beta(v_{ij} - \mu_j)}}{\sum_j e^{\beta(v_{ij} - \mu_j)}} ,$$

which establishes theorem 6. *Q.E.D.*

The result (102) is practically equivalent to equation (94), which was derived assuming utility maximisation. The equation (102) provides the demand function which, in equilibrium, must counter-balance supply. In fact the expected demand in j, D_j is given by:

$$D_j = \sum_i P_i \frac{e^{\beta(v_{ij} - \mu_j)}}{\sum_j e^{\beta(v_{ij} - \mu_j)}}$$

and since the supply function is known, i.e. the functions $\mu_j(D_j)$ are given, the equilibrium values for D_j and μ_j can be obtained as solutions to the set of equations

$$D_j = \sum_i P_i \frac{\exp\{\beta[v_{ij} - \mu_j(D_j)]\}}{\sum_j \exp\{\beta[v_{ij} - \mu_j(D_j)]\}} \; ; \; j = 1, \ldots, m. \qquad (103)$$

As far as the existence and uniqueness of the solution of (103) are concerned, it is interesting to note that they can be embedded in a concave programming problem which, besides ensuring the required properties, has a similar structure to that of problems (76) and (97) and shares the same economic interpretation. This result is established in:

Theorem 7 (Leonardi)

Assume:

$$\mu'_j \geqslant 0, \qquad \text{where} \quad \mu'_j = \frac{d\,\mu_j}{d\,D_j} \qquad (104)$$

Then the optimisation problem:

$$\max_{x,D} \; \{ \sum_{ij} x_{ij} \, (v_{ij} - \frac{1}{\beta} \ln x_{ij}) - \sum_j \int_0^{D_j} \mu_j(z)\, dz : \; \sum_j x_{ij} = P_i, \; \sum_i x_{ij} = D_j \} \qquad (105)$$

is a concave program with the solution:

$$x_{ij} = P_i \frac{e^{\beta v_{ij} - \mu_j(D_j)}}{\sum_j e^{\beta v_{ij} - \mu_j(D_j)}}, \qquad (106)$$

in which the values D_j are determined from equation (103). In addition, the value of the objective function corresponding to the optimum solution is equal, except for the additive constant, to the Total Social Benefit, that is:

$$\frac{1}{\beta} \sum_i P_i \ln \sum_j e^{\beta[v_{ij} - \mu_j(D_j)]} + \sum_j \{D_j \mu_j(D_j) - \int_0^{D_j} \mu_j(z) dz\}, \qquad (107)$$

in which

$\frac{1}{\beta} \sum_i P_i \ln \sum_j e^{\beta[v_{ij} - \mu_j(D_j)]}$ is the consumer surplus, and

$\sum_j \{D_j \mu_j(D_j) - \int_0^{D_j} \mu_j(z) dz\}$ is the producer surplus.

Proof of theorem 7

The assumption that $\mu_j' > 0$ is reasonable and implies that the supply price increases with demand.

The function

$$f_j(D) = \int_0^D \mu_j(z)\, dz$$

is convex since:

$$f_j' = \mu_j \quad , \quad f_j'' = \mu_j' \geqslant 0 \; .$$

Therefore the function

$$- \sum_j f_j(D_j)$$

is concave. The remaining term of the objective function (105) is an entropy, which, as we know, is concave in variables x_{ij}. The constraints of problem (105) are linear. Therefore it is a concave programming problem.

This means that problem (105) has a single maximum and that the Kuhn-Tuker conditions are necessary and sufficient to determine it. The Lagrangian corresponding to problem (105) is:

$$\mathcal{L} = \sum_{ij} x_{ij} \left(v_{ij} - \frac{1}{\beta} \ln x_{ij}\right) - \sum_j \int_0^{D_j} \mu_j(z)\, dz + \tag{108}$$

$$+ \sum_j \nu_i \left(P_i - \sum_j x_{ij}\right) + \sum_j \gamma_j \left(D_j - \sum_i x_{ij}\right),$$

in which ν_i and γ_j are shadow prices.

For the maximum

$$\frac{\partial \mathcal{L}}{\partial x_{ij}} = 0 \; , \quad \frac{\partial \mathcal{L}}{\partial D_j} = 0 \; ,$$

or:

$$v_{ij} - \frac{1}{\beta}(\log x_{ij} + 1) - \nu_i - \gamma_j = 0 \qquad (109)$$

$$-\mu_j(D_j) + \gamma_j = 0 \,. \qquad (110)$$

From (110) we find that the shadow price corresponding to the destination constraint is equal to the actual equilibrium market price $\mu_j(D_j)$, the same result as that obtained previously.

From (109), substituting from (110), we obtain:

$$x_{ij} = e^{-(\beta\nu_i + 1)} e^{\beta[v_{ij} - \mu_j(D_j)]}. \qquad (111)$$

From the origin constraint we have:

$$\sum_j x_{ij} = P_i \;;$$

therefore:

$$P_i = e^{-(\beta\nu_i + 1)} \sum_j e^{\beta[v_{ij} - \mu_j(D_j)]},$$

from which we obtain:

$$e^{-(\beta\nu_i + 1)} = P_i \Big/ \sum_j e^{\beta[v_{ij} - \mu_j(D_j)]} \qquad (112)$$

and substituting in (111) we obtain the result (106), which is therefore proved.

The imposition of the destination constraint

$$\sum_i x_{ij} = D_j$$

produces the equations (103), which because of the concavity of (105) determine the value D_j uniquely and consequently x_{ij} and μ_j.

The value of the objective function corresponding to the optimum solution can be obtained substituting (109) and (110) in the Lagrangian (108). We obtain from this:

$$\mathcal{L} = \frac{1}{\beta} \sum_{ij} x_{ij} + \sum_i P_i \nu_i + \sum_j D_j \mu_j(D_j) - \sum_j \int_0^{D_j} \mu_j(z)dz$$

and substituting from (106) and (112):

$$\mathcal{L} = \frac{1}{\beta} \sum_i P_i \ \ln \sum_j e^{\beta [v_{ij} - \mu_j(D_j)]} + \sum_j [D_j \mu_j(D_j) - \int_0^{D_j} \mu_j(z)dz]$$

$$- \frac{1}{\beta} \sum_i P_i \ln P_i, \tag{113}$$

which establishes the result (107).

The first term of (113) is the consumer surplus, as it is the general integral of the demand function. In fact, if:

$$\phi(\mu) = \frac{1}{\beta} \sum_i P_i \ \ln \sum_j e^{\beta(v_{ij} - \mu_j)},$$

then

$$\frac{\partial \phi}{\partial \mu_j} = - \sum_i P_i \frac{e^{\beta(v_{ij} - \mu_j)}}{\sum_j e^{\beta(v_{ij} - \mu_j)}} = - D_j \tag{114}$$

and (114), according to Hotelling (1938), is the definitory property of the consumer surplus.

Similarly, the second term of (113) is the producer surplus as it is the sum of the integrals of the supply function. In fact, if we consider the inverse function of $\mu_j(D_j)$:

$$D_j(\mu_j);$$

then, using the rule of integration by parts, we have:

$$D_j \mu_j - \int_0^{D_j} \mu_j(z)\, dz = D_j \mu_j - D_j \mu_j + \int_0^{\mu_j} D_j(z)dz =$$

$$= \int_0^{\mu_j} D_j(z)dz, \tag{115}$$

and (115) is, by definition, the producer surplus in zone j. Equation (113) therefore gives, up to an additive constant, the Total Social Benefit. *C.V.D.*.

Equation (115) makes it possible to give a dual version of (113) expressed only in terms of prices. Substituting result (115) in (113) we have:

$$D(\mu) = \frac{1}{\beta} \sum_i P_i \ln \sum_j e^{\beta(v_{ij} - \mu_j)} + \int_0^{\mu_j} D_j(z)dz - \frac{1}{\beta} \sum_i P_i \ln P_i ,$$

and the problem:

$$\min_{\mu} D(\mu)$$

which is a convex programme, dual to (105) and therefore equivalent to (105). The vanishing of the derivatives of $D(\mu)$ implies the conditions:

$$\frac{\partial D}{\partial \mu_j} = - \sum_i P_i \frac{e^{\beta(v_{ij} - \mu_j)}}{\sum_j e^{\beta(v_{ij} - \mu_j)}} + D_j(\mu_j) = 0 ,$$

which coincide with the equilibrium conditions (103).

What is important is that theorem 7 is valid independently of the fact that the supply functions $\mu_j(D_j)$ are real monetary prices. For example, (106) is formally identical to the equilibrium model of traffic assignment (discussed in this book by Smith).

Still in the field of traffic assignment, the classic problem of Beckmann, McGuire and Winsten (1956) is included in theorem 7 as a special case. In fact the idea of treating a negative externality (in this particular case, transport costs) as a supply function and of considering its integral as a "producer surplus" (even when real producers as such are not identifiable) can be attributed to these authors.

Other externalities which condition spatial markets can be treated in the same way. For example the waiting time for the assignment of a dwelling in a rationed housing market, which is discussed by Weibull in this book, could be embedded in a problem similar to (105), even though this is not explicitly considered by the author.

3.2.6. General equilibrium and spatial differentiation of prices in spatial exchange markets

So far we considered the situation in which a mobile demand consumes stock of immobile goods and services (eg. residences, road network and shopping centres). We look here briefly at the opposite case in which mobile goods are exchanged between different points in space and consumed by an immobile demand. We show that this situation can be formulated in terms which exhibit a structure quite similar to that considered in the preceding cases.

To this end, let us look at the Samuelson model (1952), discussed by Beckmann in this book. In its original version a single product is exchanged between the various points in space in which it is produced and consumed.

If:

μ_j is the price paid for one unit of product in j;

$q_j(\mu_j)$ is the net demand (i.e. demand minus production) in j, such that $q'_j < 0$;

$\mu_j(q_j)$ is the inverse function of $q_j(\mu_j)$;

x_{ij} is the export from i to j;

r_{ij} is the unit transport cost between i and j;

according to Samuelson the equilibrium pattern of prices and flows is determined by the solution to the concave programming problem:

$$\max_{q,x} \{\sum_j \int_0^{q_j} \mu_j(z)dz - \sum_{ij} r_{ij} x_{ij} : q_j = \sum_i (x_{ij} - x_{ji})\}. \tag{116}$$

The objective function of (116) is the Total Benefit (sum of consumer and producer surplus), minus transport costs, while the constraints are simple equations of import/export balance.

It is shown by Beckmann (in this book), that (116) corresponds to the dual problem:

$$\min_{\mu} \{\sum_j \int_{\mu_j}^{\infty} q_j(z)dz : \mu_j - \mu_i \leqslant r_{ij}\}. \tag{117}$$

In addition, from (116) or (117) we can derive the equilibrium conditions which have the following form:

$$\begin{aligned} &\text{if} \quad \mu_j - \mu_i < r_{ij}, \quad x_{ij} = 0 \\ &\text{if} \quad \mu_j - \mu_i = r_{ij}, \quad x_{ij} \geqslant 0\,; \end{aligned} \tag{118}$$

that is, there can be exports only when the relative advantage in terms of selling prices counter-balances the transport cost.

From the constraint of (117) we derive the fact (already mentioned previously) that transport costs are the main factors responsible for the local variation in prices, and that a reduction in transport costs consequently reduces such a variation. From the conditions (118) we can derive a rule of local specialisation which is reasonable but possibly too rigid. Indeed, supposing there are exports from i to j that is:

$$\mu_j - \mu_i = r_{ij}$$

It follows that:

$$\mu_i - \mu_j = - r_{ij} < r_{ij} \Rightarrow x_{ji} = 0$$

(assuming of course that transport costs are not negative). Hence, the exports can go in a single direction –if i exports to j, j does not export to i.

This extreme tendency can be corrected by introducing a certain dispersion of flows, as was done in (97) and (105), that is, by introducing an "entropy" in the objective function. As was shown in theorems 4 and 6 this is equivalent to the introduction of stochastic heterogeneity of preferences and choices at the micro-level. A micro-economic justification of the introduction of an entropy in problem (116) would be argued in the same way so will not be repeated here.

The result is established in the following:

Theorem 8 (Leonardi)

If $q'_j < 0$ and $\beta > 0$, the programming problem:

$$\max_{q,x} \{\sum_j \int_0^{q_j} \mu_j(z)dz - \sum_{ij} x_{ij} (r_{ij} + \frac{1}{\beta} \ln x_{ij}) : q_j = \sum_i (x_{ij} - x_{ji})\} \tag{119}$$

is concave and has the solution:

$$x_{ij} = e^{-1} e^{\beta(\mu_j - \mu_i) - r_{ij}}, \tag{120}$$

in which μ_j are determined as the solution to the equations:

$$q_j(\mu_j) = e^{-1}\sum_i\{e^{\beta[(\mu_j-\mu_i)-r_{ij}]} - e^{\beta[(\mu_i-\mu_j)]-r_{ji}]}\}. \qquad (121)$$

In addition, (119) has the corresponding dual:

$$\min_{\mu}\{\sum_j \int_{\mu_j}^{\infty} q_j(z)\,dz + \frac{e^{-1}}{\beta}\sum_{ij} e^{\beta[(\mu_j-\mu_i)-r_{ij}]}\}. \qquad (122)$$

The problem (122) is convex and:

$\sum_j \int_{\mu_j}^{\infty} q_j(z)\,dz$ is the consumer surplus, and

$\frac{e^{-1}}{\beta}\sum_{ij} e^{\beta[(\mu_j-\mu_i)-r_{ij}]}$ is the producer surplus.

Proof of theorem 8.

The functions:

$$F_j(q) = \int_0^q \mu_j(z)dz$$

are concave since:

$$F'_j(q) = \mu_j(q), \quad F''_j(q) = \mu'_j(q) \leqslant 0.$$

The second term of the objective function of (119) is an entropy and therefore a concave function. The constraints are linear, therefore (119) is concave. The Kuhn-Tucker conditions are consequently necessary and sufficient to determine the unique maximum. The Lagrangian associated with (119) is:

$$\mathcal{L} = \sum_j \int_0^{q_j} \mu_j(z)dz - \sum_{ij} x_{ij}(r_{ij} + \frac{1}{\beta}\ln x_{ij}) + \sum_{ij} x_{ij}(\gamma_j - \gamma_i) - \sum_j q_j\gamma_j, \qquad (123)$$

in which γ_j are shadow prices associated with the constraints. The vanishing of the derivatives of the Lagrangian with respect to q_j and x_{ij} implies:

$$\frac{\partial\mathcal{L}}{\partial q_j} = \mu_j(q_j) - \gamma_j = 0 \qquad (124)$$

$$\frac{\partial\mathcal{L}}{\partial x_{ij}} = -r_{ij} - \frac{1}{\beta}(\ln x_{ij} + 1) + (\gamma_j - \gamma_i) = 0. \qquad (125)$$

Equation (124) implies that shadow prices are equal to local equilibrium prices, while (125) when solved with respect to x_{ij} after substituting from (124), gives the result (120) which is therefore proved. The result (121) is easily obtained by substituting (120) in the constraints of import-export balance.

To obtain the dual of (119) we substitute (124) and (125) in (123). After some steps we obtain:

$$\mathcal{L} = \sum_j \int_0^{q_j} \mu_j(z)dz - \sum_j \mu_j q_j + \frac{e^{-1}}{\beta} \sum_{ij} e^{\beta[(\mu_j - \mu_i) - r_{ij}]}$$

and using the equality, which can be shown integrating by parts:

$$\int_o^{q_j} \mu_j(z)dz = \int_{\mu_j}^{\infty} q_j(z)dz + q_j \mu_j ,$$

we obtain the dual problem:

$$\min_{\mu} \{\sum_j \int_{\mu_j}^{\infty} q_j(z)dz + \frac{e^{-1}}{\beta} \sum_{ij} e^{\beta[(\mu_j - \mu_i) - r_{ij}]}\},$$

which is identical to (122). Problem (122) is convex, since it is the dual of a concave problem. The first term of its objective function is by definition the consumer surplus. To show that the second term is the producer surplus, it is sufficient to show that it is the general integral of the net supply (imports minus exports) in each zone. We can define the function:

$$G(\mu) = \frac{e^{-1}}{\beta} \sum_{ij} e^{\beta[(\mu_j - \mu_i) - r_{ij}]};$$

therefore the following property holds:

$$\frac{\partial G}{\partial \mu_j} = e^{-1} \sum_i \{e^{\beta[(\mu_j - \mu_i) - r_{ij}]} - e^{\beta[(\mu_i - \mu_j) - r_{ji}]}\},$$

that is, from (120):

$$\frac{\partial G}{\partial \mu_j} = \sum_i (x_{ij} - x_{ji}).$$

Therefore the derivatives of G are actually the net supply in each zone and G is the producer surplus. *C.V.D.*.

The structure of (119) is formally analogous but in fact opposite to that of problem (105).

In the latter, dispersion is introduced into consumer flows (which are taken to be mobile) and the supply function is considered given. In (119) dispersion is introduced in commodity flows (considered to be mobile), while consumers are taken to be immobile and the demand function given.

Solution (120) is clearly more flexible than the classic conditions (118) since all flows, in all directions, are in general non zero and we can have exports even when the relative advantage in selling prices does not counterbalance the transport cost. Export flows are nevertheless increasing with the difference:

$$(\mu_j - \mu_i) - r_{ij} = (\mu_j - r_{ij}) - \mu_i ,$$

which is the difference between net profit deriving from the sale of one unit of product in j, given by the price in j minus the transport cost, and the profit deriving from the same sale in i without transport cost.

In general transport costs are not the sole costs responsible for local differentiation in prices.

Even when we set $r_{ij} = 0$, $\forall i,j$ in (121), in general they will not have a solution of the type $\mu_j = \mu$ $\forall j$, since this would imply that:

$$q_j(\mu) = 0 \quad , \quad \forall j$$

and in general net demand functions are different in each j. However, in the special case when they do not vary locally, that is:

$$q_j(\mu) = q(\mu) \quad , \quad \forall j ,$$

the price wich satisfies the equation:

$$q(\mu) = 0$$

also satisfies (121) with $r_{ij} = 0$ and is therefore a solution of problem (119).

In this case there is a uniform price, net demand disappears in all points (i.e. consumption and production are equalised locally), but this does not imply there are no exports.

In fact if we set $\mu_j = \mu$ and $r_{ij} = 0$ in (120) we obtain:

$$x_{ij} = e^{-1} > 0 \ .$$

3.2.7. A short note on price equilibrium in multi-sector systems

The equilibrium conditions we have looked at up to now concern a single commodity and are essentially neoclassic as they can be reduced to the total benefit maximisation principle.

Here we examine a case in which the neoclassic paradigm seems to lead to a paradox. A more detailed discussion of the subject is given in this book by Sheppard. We look at some rather simpler examples.

Consider a multi-sector non-spatial system with linear technology, defined by the following input-output coefficients:

a_{ij} quantity of product i used to produce one unit of product j;

Define further:

x_j total production in sector j;

μ_j price of product j;

$q_j(\mu_j)$ final demand of product j.

In the absence of stock accumulations, the balance equations must be:

$$x_i = \sum_j a_{ij} x_j + q_i(\mu_i) ; \tag{126}$$

that is, the production of each sector must be equal to the sum of the demand of other sectors and the final demand. Using the same reasoning as in 3.2.6. we can try to determine the level of price and quantity in equilibrium by maximising total benefit, given by:

$$\sum_i \int_0^{q_j} \mu_i(z)dz , \tag{127}$$

in which $\mu_i(z)$ is the inverse function of q_i, subject to the constraints (126).

The corresponding Lagrangian is:

$$\mathcal{L} = \sum_i \int_0^{q_j} \mu_i(z)\,dz + \sum_i \gamma_i \left[x_i - \sum_j a_{ij} x_j - q_i \right] \tag{128}$$

in which the γ_i are shadow prices associated with the constraints (126). The vanishing of derivatives of (128) implies that:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \mu_i(q_i) - \gamma_i = 0 \tag{129}$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = \gamma_i - \sum_j \gamma_j a_{ji} = 0 . \tag{130}$$

Equation (129) implies, as before, that the shadow prices equal actual market prices.

Equation (130) has a non-zero solution only if the input-output matrix has an eigenvalue equal to 1 (in this case the price are the associated right eigenvector). We known however from economic theory (cf.: Sheppard, in this book) that for a real input-output matrix the maximum eigenvalue is less than 1. The only possible solution for (130) is therefore:

$$\gamma_i = 0, \quad \forall\, i,$$

which clearly makes no sense in any real system.

The degenerate nature of such a solution leads us naturally to a generalisation of the constraints (126).

We assume that there is a stock accumulation directly proportional to total demand, such that:

$$x_i = (1+\alpha)\,[\sum_j a_{ij}\, x_j + q_i(\mu_i)] , \tag{131}$$

in which $\alpha > 0$ is an accumulation rate (at present unknown) which is equal for all sectors. The maximisation of (127) subject to the constraints (128) gives the Lagrangian:

$$\mathcal{L} = \sum_i \int_0^{q_i} \mu_i(z)\, dz + \sum_i \gamma_i\, [x_i - (1+\alpha)\; (\sum_j a_{ij}\, x_j + q_i)] \tag{132}$$

and the conditions:

$$\frac{\partial \mathcal{L}}{\partial q_i} = \mu_i(q_i) - (1+\alpha)\, \gamma_i = 0 \tag{133}$$

$$\frac{\partial \mathcal{L}}{\partial x_i} = \gamma_i - \sum_j (1+\alpha)\, \gamma_j\, a_{ji} = 0 . \tag{134}$$

Substituting (133) in (134) we obtain:

$$\mu_i = (1+\alpha) \sum_j \mu_j a_{ji}$$

or in matrix form:

$$\mu = (1+\alpha) \mu A, \tag{135}$$

in which

$$\mu = \{\mu_i\}$$

$$A = \{a_{ij}\}.$$

(135) clearly has the same structure as the price formation equation proposed by several marxian economists such as Sraffa (1960), Morishima (1973) and Sheppard (in this book).

The unknown α can be determined by imposing the condition that prices are non-negative. One solution, obtained previously, but of no practical interest, in that is which prices are zero everywhere.

Another solution is provided by matrix analysis. We know that the only non-negative eigenvector associated with a positive matrix is that corresponding to the maximum eigenvalue. If we define:

$$\lambda = 1/(1+\alpha),$$

equation (135) can be rewritten as

$$\lambda\mu = \mu A,$$

where λ must be the maximum eigenvalue of A. As stated before, we know that:

$$\lambda < 1,$$

and this implies that $\alpha > 0$, as required.

Equation (135) shows that α can be interpreted not only as a stock accumulation rate, but also as profit rate. In fact, if $\alpha = 0$, prices would be equal to production costs and as we have shown this policy would give us, in equilibrium, zero prices everywhere.

What is perhaps surprising is that the result (135) was obtained by introducing a slight modification (i.e. the stock accumulation rate) in an essentially neoclassic problem, much as total benefit maximisation, a concept obviously ignored in Marxian economics.

3.3. *The technological structure of inter-sector transactions and production and consumption mechanisms*

3.3.1. Introduction

In this section we analyse the relationships between location and transport, considering them as consequences of the technological structure underlying production, consumption and the transport of goods. In this book we find two completely opposite approaches to the problem — the neo-classic approach of Beckmann and the Marxian approach of Sheppard — and a number of other contributions which deal with the question of technological structure less explicitly, but, in treating the entropic dispersion of neo-classic equilibrium in various ways form a kind of "bridge" between the two extremes.

This analysis is divided into three parts. The first covers the neo-classic approach and refers principally, as explained above, to Beckmann's work, although including mention of other authors not necessarily from this book. The second concerns studies defined above as involving entropic dispersion. The third discusses the Marxian approach. In the concluding section we examine a number of important questions concerning technological structure not explicitly deal with in the other chapters of this book. These questions and in particular the problems involved in the innovation process constitute an important subject for future research in each of the three approaches mentioned above (neoclassic, entropic and Marxian).

Naturally in dealing with the relationship between location, transport and mechanisms of production and consumption we cannot ignore the close interrelationships between quantities (produced and consumed) and the relative prices. In this respect the analysis carried out here is closely linked to that in 3.2. which focused on the mechanisms for the formation and spatial differentiation of prices.

3.3.2. The neoclassic approach

The standard theory of transport and location was developed during the 1950's with the work of authoritative economists such as Samuelson (1952) and Beckmann (1968). Paraphrasing Anderson (1983), it can be said that the neo-classic model dealing with the relationships between location, trade and transport has the following basic elements:

– a set of goods located in a set of regions;
– a predefined type of production technology for the above goods described by a typical neoclassic production function (i.e. concave continuous and at least twice differentiable) with different values from region to region;
– a transport system connecting each region with all the other regions with a demand proportional to the volume of goods to be transferred from one region to another and a supply defined by a typical neoclassic production function;
– a function of total welfare (which is continuous concave and at least twice differentiable) which when optimised provides a description of the desired state of the system that is, a state of global stable equilibrium).

By defining the above elements in different ways, different versions of the basic model are obtained, some of which are discussed by Beckmann in this book. For example Samuelson gives the following formulation:

$$\max_{q,x} \left[\sum_j \int_0^{q_j} \mu_j(z)dz - \sum_{ij} r_{ij} x_{ij} \right], \tag{136}$$

subject to the constraint:

$$\sum_i (x_{ij} - x_{ji}) = q_j \ , \tag{137}$$

where:

q_j is the net quantity of the good demanded in j;
μ_j is the price in j of the good (function of q_j);
r_{ij} is the cost of transport from i to j;
x_{ij} is the flow of goods from i to j.

According to the general rule, the solution to the programming problem (136) with the constraint (137) is a solution of stable equilibrium given by the conditions:

$$x_{ij} > 0 \quad \text{if} \quad p_j - p_i = r_{ij}$$
$$x_{ij} = 0 \quad \text{if} \quad p_j - p_i < r_{ij}. \tag{138}$$

The problem of location and transport can obviously be analysed in the context of continuous space as well as discrete space.

Samuelson's model has an elegant continuous version formulated by Beckmann (1952). In this version the constraint (137) is expressed as:

$$\operatorname{div} \Phi + q = 0 , \tag{139}$$

which states that the divergence of the flow of goods is equal to the net quantity demanded. The maximising welfare function (136) becomes the following variational problem:

$$\min_{\Phi} \int_{R} k|\Phi| \, dR \tag{140}$$

(where k is the unit cost of transport in a given point of the region R), whose solution is:

$$k \frac{\Phi}{|\Phi|} = \operatorname{grad} p \qquad \text{if} \quad \Phi \neq 0 , \tag{141}$$

which states that the direction of the flows is given by the gradient of a potential function p, which represents prices.

The advantage of using continuous space is that under the usual highly simplified hypotheses (of homogeneous and isotropic space) it is often possible to analyse spatial structures deriving from these models. A great deal of work has been achieved in this aspect by Puu (1979a). We should like to make special mention of his work on the spatial organisation of structurally stable flows. He bases his reasoning on the fact that commodity flows are defined [see (141)] by a potential function which means that we can apply to them structural stability considerations of the type found in catastrophe theory. On these bases, we can deduce that the flows actually observed are those which are structurally stable and not completely altered by small disturbances.

It is also shown that as far as critical points in the flow space are concerned, the most probable to occur in reality are isolated critical points (i.e. isolated nodes and saddles). Putting together these two observations and using theorems relating to the general properties of differential equations, Puu comes to the conclusion that the flow structure which links isolated critical points is a square grid pattern whereas the triangular grid, which would result from the spatial organisation pattern described by Christaller and Lösch, is unstable.

This important result clearly deserves more detailed examination in future studies, as well as the relationship between the perturbed flows considered by Puu and the flows described by the entropic dispersion of the neo-classic optimum (which could also be viewed as disturbed flows).

3.3.3. Entropic dispersion

In 3.3.2. we refered to the production function as one of the central elements in the location of economic activities. This function links the output of one industrial sector in a given location with its input:

$$y_{j*}^{s*g} = y_{j*}^{s*g}(\{x_{*j}^{*om}\}) \tag{142}$$

where

x_{ij}^{rsm} is the flow of goods or services of type m, used by a firm s located in j and produced by a firm r located in i;

y_{jk}^{svg} is the flow of goods or services of type g, produced by a firm s located in j and used by a firm v located in k;

(the star stands by the sum of the indices considered) taking a neoclassic approach this function must satisfy a certain set of properties. Among these properties is the existence of a defined and constant production technology which can be described by a matrix of technological coefficients. This technology can be introduced into the production function using the following kind of formula:

$$x_{ij}^{rsm} = q_{ij}^{rsm}\, a^{mg}\, y_{j*}^{s*g}\,, \tag{143}$$

where

a^{mg} is the quantity of good m required to produce one unit of product g;

q_{ij}^{rsm} is the quantity of good m used by a firm s in j coming from another firm r in i.

The matrix a^{mg} in (143) contains information on the degree of interdependence of production and the matrix q_{ij}^{rsm} contains information on spatial interdependence. There is an important difference in the treatment of these two aspects. While the technological coefficient matrix is a structure which is descriptive of the process of intersector interactions, the matrix of spatial interdependence coefficients is an optimum structure which is in certain respects unrealistic. If for example $q_{ij}^{rsm} > 0 \Rightarrow q_{ji}^{rsm} = 0$, this means that the flows are always one-directional, which is rarely the case – in reality, flows are usually two-directional. The introduction of a term of entropic dispersion in the neoclassic optimum function by Wilson (1970a) provided a description of the structure of spatial relations more like that of inter sector relations. The concept of entropic dispersion is also found in intersector transaction analysis. A very common method for updating input/output matrices, the RAS method of bi-proportional adjustment (Bacharach, 1970) can be considered an estimating technique using entropy-maximisation. In the same way studies of economic dominance (Lantner, 1974) can be seen as a search for the latent "optimal" structure in a given sectoral interdependence matrix.

Various theoretical justifications have been given of the entropic structure of spatial (and sector) relations in addition to the socio-physical justification of Wilson – random utility, consumer surplus etc.. Smith in this book gives a further one, that of cost efficiency. This is particularly interesting as, more than the others, it provides a clear behavioural base to structures of entropically dispersed spatial interaction. Let us consider the objective function:

$$\min_{(T_{ij})} \; C\,(T_{ij}) - \frac{1}{\theta}\, H\,(T_{ij}) \tag{144}$$

where T_{ij} are the flows, H is the flow entropy and C is the overall cost of flows. We can recognise in this function either the problem of entropy-maximising associated with a transport cost constraint or a problem of minimisation of transport costs (Beckmann) with an entropic dispersion. Smith shows that it can be derived from a principle of cost efficiency which postulates that:

$$\bar{C} \leqslant \bar{C}' \Rightarrow P \geqslant P' \, , \tag{145}$$

that is, the average costs of two flows distributions are in inverse relation to the probabilities of the two distributions.

Given the above we can add that Wilson in his contribution to this book formulates a general model of the interrelationships between transport and location with a production function of the kind in (143) which is consistent with an entropic dispersion of commodity flows. Using the same symbols as above this gives:

$$\underset{\{x_{ij}^{rsm},\ y_{ik}^{svg}\}}{\text{Max}}\ Z = \sum_{js} (D_j^s - C_j^s)\ , \tag{146}$$

which is a mathematical programming problem in which D_j^s are revenues from the output of the industrial units svj and C_j^s the costs of the relative input. (146) must of course be resolved taking into account specific constraints on the matrices x_{ij}^{rsm} and y_{ik}^{svg} which also describe the production technology and the entropic structure of spatial interaction.

The formulation of operative version of (146) constitutes one of the most promising directions for future research.

In conclusion we point out that Wilson in this book proposes a dynamic version of model (146), this dynamic character deriving from the disequilibrium between costs and revenue:

$$\dot{y}_{j*}^{s*g} = \varepsilon^{sg} (D_j^{sg} - C_j^{sg})\ , \tag{147}$$

where D_j^{sg} and C_j^{sg} are respectively the revenue and costs functions (in general non linears) of y_{j*}^{s*g}.

3.3.4. The Marxian approach

To introduce this section we shall first of all look briefly again at the mathematical programming problem (146). It is clear that given the general terms in which the problem is formulated we do not know whether it is convex or concave, in other words we have no idea whether the system of locations and transport considered will have one or more solutions or whether they will be stable or unstable.

It should be added however that when, as suggested in 3.2.2., we make the usual neoclassic assumptions, the system will have in fact one stable solution.

The question that we must ask is therefore if the consideration of an entropic dispersion in commodity flows will modify the neo-classic results. It seems that we are not able at present to come to any definite conclusions about this. There is on the one hand the result of Macgill (1977b) which shows that an input-output model which is spatially disaggregate with entropic spatial relations has a stable equilibrium solution. On the other hand there is a notable similarity between model (147) and the model of service location of Harris and Wilson (1978) which leads us to believe that for particular forms of non-linearity in spatial interaction we may have solutions of multiple or instable equilibrium or no solutions at all. It remains to be seen in this second case whether the hypotheses which lead to these non-linearities are in contrast with the neoclassic assumptions or not.

It is in this context, the discussion of the existence of equilibrium in the location transport system, that Sheppard's contribution to this book is particularly relevant. He begins with the extension to the spatialised economy (i.e. the location-transport system) of the criticisms which the neo-Ricardian economists, and in particular Sraffa (1960), made of Walras' general economic equilibrium.

In this way Sheppard develops a descriptive Marxian model of multi-sector activity location, the spatial structure of commodity prices and the value of labour

$$p = (1+r)\ p\ A(p), \tag{148}$$

where:

p is the price vector;

r is the profit rate;

A is the input-output matrix extended to include wages as a production input. It is a function of p; and its spatial structure is described by an entropy-gravitational type expression (where spatial interaction is a function of p and transport costs).

Sheppard then shows that given the spatial structure of the production (148) will have an equilibrium solution, which is now however the neo-classic Walrasian equilibrium as previously assumed, but quite different, because it depends on a set of social factors from Marxian and Ricardian analysis (competitive capitalism, conflicting social classes, exploitation of the labour force and political power) which are considered in the production structure.

As far as the dynamic aspects of the spatial equilibrium of Sheppard's model are concerned, given that unstable spatial structures of production may occur over time, there may also be instability in geographical structures i.e. in the locations of multisectoral activities and their spatial interrelations.

The general conclusion is that anarchy in competitive capitalism cannot produce a market equilibrium (which is the opposite of the standard neoclassic conclusion) without government intervention.

A further point of great interest in Sheppard's chapter is that the system of pricing determined by (148), that is, the Marxian pricing system, is shown to be equivalent to that obtained from the consideration of "potentials" associated with the spatial interaction between economic activities:

$$p = i\,U \qquad (149)$$

where:

- i is a vector of ones;
- U is the matrix of potentials associated with the iterative process of goods transfer of which one step is described by the matrix A considered earlier.

In other words the Marxian system of pricing is equivalent, approximatively speaking, to the pricing system determined by the respective geographical accessibilities of the different activities.

This result means that the geographical and economic approaches are virtually the same, which is in complete contrast with the conventional socio-economic theory (which tends to make spatial interaction depend on economic quantities) and opens up a highly promising area for future research.

3.3.5. Some remarks on technological structures

There is one aspect of the technological structure of production – innovation in technology – which at present is the subject of great interest, prompted by the stimulus of what is happening in the real world.

It is an aspect which has been relatively neglected in the contributions to this book. Only Sheppard's chapter considers the consequence of variations in the matrix of sectoral and spatial interdependencies (extended to consider also the labour force) and deduces from them the instability of the growth of the spatialised economy.

The question of innovation deserves to be looked at in more detail, especially in its effects on location and transport.

Several authors provide us with possible points from which to begin. We could for example attempt to apply Wilson's ideas on the growth and evolution of service infra-structure (Wilson 1981a) or the work of the Brussels school (Allen et *al.* 1978) on urban morphogenetic processes to the problem of technological innovation.

Alternatively, following the example of Anderson (1983) we could try to extend to the location-transport system the results obtained from non spatial economic analysis, where the problem of changing production structures is related to a linear combination of intersector matrices each of which represents a different type of technology. In this context the fact that the different production technologies are in competition with each other must be taken into account. Sonis (1983) makes some contributions to the analysis of this aspect of the problem.

The problem of technological innovation is also closely connected with the question of diffusion of information especially when we are dealing with a spatial context. Studies in the field of information diffusion, for example Ralston, 1983, who considers the dynamics of communications, could be usefully applied to the problem.

Last of all technological innovation is linked to the problem of investment in research and development. Here the work of Nijkamp (1983) for example could play a useful part in the analysis.

4. The most promising directions for future research

In this final section we look at the emerging areas of research which we feel deserve to be given a certain priority and on which efforts need to be concentrated in the future if significant progress is to be made.

These areas, all relating of course to location-transport systems, are the following:

1. the analysis of the dynamic structure;
2. the evaluation and testing of performance;
3. the relationship between individual and collective behaviour.

We will now briefly examine each of these aspects considering the need for:

a. an identification of the problems (in the light of the current state-of-the-art) and establishment of a general framework for the study programme;
b. the developments necessary in methodology;
c. the organisation of the research programme.

Taking the areas above in turn.

The analysis of the dynamic structure:

a. it emerges from the survey of the current situation that dynamic models of stock do exist but that they tend to neglect the dynamics of flow (Harris and Wilson, 1978, Allen and Sanglier, 1979a, Wilson and Clarke, 1979, Allen and Sanglier, 1981a, Wilson, 1981b, Clarke and Wilson, 1983a, Lombardo and Rabino, 1983a, Wilson, in this book). Vice-versa, there are dynamic models of flows but these tend to neglect the dynamics of stock (Leonardi and Campisi, 1981, Weidlich and Haag, 1983, de Palma and Lefèvre, in this book, Leonardi, in this book, Wiebull in this book).

   As far as the interaction between the dynamics of stock and the dynamics of flows is concerned, the present state of affairs is far less satisfactory, even though some interesting attempts have been made in some sectors (see, Snickars, 1978). It appears that from the point of view of application the main effort needs to be concentrated on the development of theories and models which deal dynamically with both stock and flows for certain urban subsystems (eg. the residential subsystem) and for the urban system as a whole. It also seems necessary to intensify study efforts on the construction of models which take into account the dynamics of the transport infrastructure. There are no models of the dynamics of transport in stock-flow interactions and they receive just a brief mention in Wilson (1983).

   A further subject which should not be overlooked in this context is the analysis of the relationship between system dynamics and innovation in the underlying technological structure. This is an involved problem which includes the question of the diffusion of information (Ralston, 1983), competition between alternative technologies (Sonis, 1983) and the role of technological research in the productive apparatus (Nijkamp, 1983);
b. from the methodological point of view, one fundamental aspect which needs to be developed is the analysis of non-linearities (especially the phenomena of sinergetics). A further aspect is the analysis of stochastic components (see de Palma and Lefèvre, for the implications in dynamic analysis and Smith for the

implications on the analysis of individual behaviour, both in this book);

c. a future research programme should include the following phases:
   - a detailed examination of the theoretical structure of models, particularly those concerning the interaction between the various subsystems and between stock and flows;
   - the preparation of the statistical methods and computational tools necessary for empirical testing and implementation of the models;
   - a comparison of applications of models in different urban systems.

The evaluation and testing of performance:

a. it emerges from the review of the present state-of-the-art that there are well developed techniques of evaluation and optimisation for static equilibrium systems (Coelho and Williams, 1978, Wilson et *al*., 1981, Beaumont, Colorni and Voogd, all in this book).
   For dynamic systems the situation is satisfactory for aggregated economic systems but inadequate for spatially disaggregate systems (although an attempt to apply optimising and dynamic control methods to spatially disaggregate models has been made by Fujita, 1978);

b. from the methodological point of view, it should be noted that the problem of controlling a dynamic system is qualitatively different from the problem of optimisation of a static system in that it involves the use of optimisation techniques (dynamic programming and optimum control) but also poses problems of structural stability adaptability and self-regulation.
   The following tasks therefore need to be undertaken:
   - a more complete analysis of the use of dynamic optimisation techniques;
   - an introduction of adaptive and self-regulating mechanisms in location-transport systems;
   - the finding of suitable mechanisms and tools for the control of structural changes;

c. the research programme should be divided into the following phases:
   - a detailed examination of the problems delineated above in b. as they relate to location-transport systems;
   - the development of suitable techniques for simulating the control mechanisms for dynamic systems, in particular for simulating the performance of adaptive and self-regulation mechanisms.

The relationship between individual and collective behaviour:

a. from the review of the current situation it emerges that the various microscopic theories (most of the work of Beckmann and Papageorgiou, including their contributions to this book) and macroscopic theories (most of the work of Wilson and Sheppard including their contributions to this book) regarding the behaviour of the actors in the urban system are well represented.

   There is not however an equally thorough treatment of the interactions between the two levels, although de Palma, Leonardi and Smith have recently carried out work on this, as can be seen in this book;

b. from the methodological point of view, there is a need to examine:
   - the sensitivity of the behaviour of a macro-level system to different hypotheses concerning micro-level behaviour;
   - the effects on micro-level behaviour of constraints and interactions at macro-level;
   - the role of the time dimension in micro-level behaviour (processes of adaptive learning);

c. as far as future research is concerned, the following are necessary:
   - a detailed analysis of the theories relating to b. (particularly the first point);
   - empirical testing of the theories relating to b. (particularly the second point).

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WORKING PAPERS

*1 "Un modello urbano a larga scala per l'area metropolitana di Torino", *gennaio 1981*

*2 "Metodologie per la pianificazione dei parchi regionali", *gennaio 1981*

*3 "A Large Scale Model for Turin Metropolitan Area", *maggio 1981*

4 "An Application to the Ticino Valley Park of a Mathematical Model to Analyse the Visitors Behaviour", *luglio 1981*

5 "Applicazione al parco naturale della Valle del Ticino di un modello per l'analisi del comportamento degli utenti: la calibrazione del modello", *settembre 1981*

6 "Applicazione al parco naturale della Valle del Ticino di un modello per l'analisi del comportamento degli utenti: l'uso del modello", *settembre 1981*

*7 "Un'analisi delle relazioni esistenti tra superficie agricola utilizzata ed alcune principali grandezze economiche in un gruppo di aziende agricole piemontesi al 1963 e al 1979", *settembre 1981*

8 "Localizzazione ottimale dei servizi pubblici, con esperimenti sulle scuole dell'area torinese", *settembre 1981*

9 "La calibrazione di un modello a larga scala per l'area metropolitana di Torino", *ottobre 1981*

10 "Applicazione al parco naturale della Valle del Ticino di un modello per l'analisi del comportamento degli utenti: l'individuazione di un indicatore di beneficio per gli utenti ed una analisi di sensitività su alcuni parametri fondamentali", *ottobre 1981*

11 "La pianificazione dell'uso ricreativo di aree naturali: il caso del parco della Valle del Ticino", *novembre 1981*

*12 "The Recreational Planning of Country Parks: the Case Study of the Ticino Valley Park", *marzo 1982*

13 "Alcuni aspetti della calibrazione di un modello dinamico spazializzato: il caso del modello dell'area metropolitana torinese", *settembre 1982*

*14 "L'applicazione di un modello dinamico a larga scala per l'area metropolitana di Torino: la calibrazione", *novembre 1982*

15 "Modello commerciale Piemonte", *novembre 1982*

16 "Resource allocation in multi-level spatial health care systems: benefit maximisation", *dicembre 1982*

17 "Relazione sulla struttura e sulla dinamica del settore elettromeccanico piemontese", *dicembre 1982*

18 "Evoluzione della finanza locale in Piemonte e in Italia 1977 - 1981", *febbraio 1983*

19 "Un metodo per l'analisi di scenari multidimensionali in ordine alle relazioni tra domanda di trasporto e variabili strutturali dei sistemi economici e territoriali", *febbraio 1983*

20 "Modello commerciale Piemonte", *marzo 1983*

21 "Calibrating the residential location submodel of the simulation moder for the Turin metropolitan area", *giugno 1983*

22 "Dinamiche spaziali dell'area metropolitana di Torino negli ultimi tre decenni", *giugno 1983*

23 "Struttura economica delle imprese del dettaglio alimentare in Piemonte – prime valutazioni", *luglio 1983*

24 "The dynamics of Turin metropolitan area: a model for the analysis of the processes and for the policy evaluation", *agosto 1983*

25 "Un'analisi, con il modello RAMOS, della struttura spaziale del servizio sanitario regionale: il caso del Piemonte", *settembre 1983*

26 "Manuale per l'uso del modello RAMOS (Resource Allocation Model Over Space)", *settembre 1983*

27 "The spatial dynamics of the Turin metropolitan area: an analysis of the last three decades", *ottobre 1983*

28 "Un modello del sistema urbano di Torino: alcune valutazioni di un'esperienza modellistica", *novembre 1983*

29 "Il conto economico dei comparti manifatturieri piemontesi, 1980 – Elaborazioni su dati rilevati dall'ISTAT sul Prodotto Lordo delle imprese manifatturiere con sede sociale in Piemonte", *novembre 1983*

30 "Interrelazioni tra localizzazioni e trasporti: stato dell'arte e possibili linee di sviluppo futuro", *gennaio 1984*

31 "Fondamenti per un approccio unificante all'analisi del comportamento della domanda in un sistema localizzazioni-trasporti", *gennaio 1984*

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